Comments on the motion of magnetic field lines

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Belcher and Olbert recently showed that the concept of the motion of magnetic field lines can be helpful in teaching classical electromagnetism. Although this concept holds in many situations, it has important limitations. It is shown that the most common definition, $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$, which is the one used by Belcher and Olbert, is not appropriate when an electrostatic field is present, unless the field satisfies special conditions. In an infinitely conducting medium where the electric field has no component parallel to the magnetic field, $\mathbf{E}\times\mathbf{B}/B^2$ is still a meaningful definition of the motion of magnetic field lines (which follow the plasma motion as if "frozen-in"). It used to be assumed that space plasmas could be treated as infinitely conducting and therefore the concept of magnetic field line motion was used extensively. But local nonvanishing values of $\mathbf{E}\cdot\mathbf{B}$ can "cut" magnetic field lines and invalidate the frozen-in condition.
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I. INTRODUCTION

Although the concept of the motion of magnetic field lines is inherently meaningless for reasons discussed by Feynman *et al.*,[1] it can sometimes be usefully defined as $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$, where $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, and $\mathbf{v}$ is the local velocity of the magnetic field line. Belcher and Olbert[2] have shown several examples of the use of the concept of magnetic field line motion that apply to vacuum fields. In Sec. II the vacuum case is discussed and it is shown that this definition of field line motion breaks down in the presence of an electrostatic field. It remains valid under certain conditions, such as in the absence of electric field components parallel to the magnetic field. For brevity the term "parallel electric field" will be used for a nonvanishing electric field component along the magnetic field.

The concept of magnetic field line motion has been used extensively in space plasma physics. In the context of Alfvén's discovery of what is now called Alfvén waves, he noted[3] that "the matter of the medium in which the waves travel appear to be 'fastened' to the lines of force;" that is, the magnetic field lines appear to move with the matter. This behavior is now described as a "frozen-in magnetic field."[4] In the late 1950s it was generally believed that the parallel electric field was everywhere zero in space plasmas, because the essentially unimpeded motion of electrons and ions along the magnetic field in nearly collisionless space plasmas would cause any such components to be "shorted out," that is, extinguished by the rearrangement of electrons and ions in the plasma, and a state of frozen-in magnetic field would prevail. Therefore, little attention was given to Alfvén's suggestion[5] that magnetic-field aligned electric fields exist above the ionosphere. As soon as *in situ* measurements were made in the space plasma, the first indications in support of Alfvén's suggestion were found,[6] and the existence and importance of magnetic-field aligned electric fields in space plasmas is now generally accepted. Therefore, the usefulness of the concept of moving field lines is limited, and as emphasized by Alfvén,[5] it can even be misleading.

II. THE VACUUM CASE

To illustrate the limited applicability of the concept of field line motion, consider the example of a magnetic dipole field in the presence of a homogeneous electric field parallel to the dipole axis. We have in spherical coordinates r, θ, φ:

$$B_r = B_p \cos\theta, \quad B_\theta = 0.5 B_p \sin\theta, \quad B_\varphi = 0, \tag{1}$$

and

$$E_r = E_o \cos\theta, \quad E_\theta = -E_o \sin\theta. \tag{2}$$

For the same definition as used by Belcher and Olbert,[2] the local "field line velocity" $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$ is given by

$$v_r = v_\theta = 0, \quad v_\varphi = (6E_o/B_p)\sin\theta\cos\theta/(1+3\cos^2\theta) \tag{3}$$

which implies an angular velocity $\omega=(6E_o/B_p r)\cos\theta/(1+3\cos^2\theta)$. Equation (3) implies that different parts of a magnetic field line would rotate at an angular velocity that not only varies along the field line but even changes sign. Thus $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$ describes a velocity distribution along the field line that would continually distort it, while in reality the magnetic field remains undistorted, and each of its field lines remains confined to a plane φ=constant.

Although this example is simple, we can imagine other configurations where the local field line velocity $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$ does not describe the real evolution of the field line. Indeed, for it to do so, it would need to have the property that different segments of a field line will remain segments of a common field line in the real magnetic field. It has been shown by Newcomb[7] that $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$ satisfies this requirement, and thus is a meaningful definition of field line velocity if

$$\mathbf{B}\times \text{curl}[\mathbf{B}(\mathbf{E}\cdot\mathbf{B}/B^2)] = 0. \tag{4}$$

This condition is trivially fulfilled if the parallel electric field is zero everywhere.

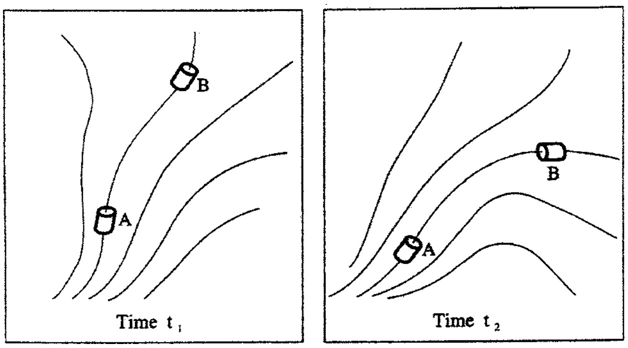

Fig. 1. Frozen-in magnetic field. Two elements of plasma that are on a common magnetic field line at any one time, t_1, will be on a common magnetic field line at any other time, t_2.

III. INFINITELY CONDUCTING FLUID

In an infinitely conducting medium, where infinite electrical conductivity ensures that "parallel electric fields" are everywhere zero, a "frozen in magnetic field" is a simple consequence of ideal magnetohydrodynamics. More rigorously, the state of frozen-in magnetic field lines is one where any two elements of the medium that are at one instant on a common magnetic field line will be on a common magnetic field line at any other instant (see Fig. 1). This definition does not imply any concept of motion of magnetic field lines. However, if this frozen-in condition holds, elements of the medium can be used to "label" the magnetic field lines and thus define their "motion." When ideal magnetohydrodynamics is applicable, the frozen field condition is an attractive tool for simplifying magnetohydrodynamic problems.

IV. SPACE PLASMA

Almost all of the known matter in our universe is in a state of magnetized plasma. In some regions, such as in the sun and stars, the high electrical conductivity makes magnetohydrodynamics an acceptable approximation and therefore magnetic field line motion is a useful concept. In the Earth's magnetosphere, solar and stellar winds, and in interstellar and intergalactic space the plasma is very nearly collisionless, that is, the density is so low that collisions between particles are very infrequent and can be disregarded. The electrical conductivity is very anisotropic. As electrons and ions move almost unimpeded along the magnetic field, it was commonly assumed that the electric field component along the magnetic field would be zero, because the unimpeded motion of charge carriers along the magnetic field would "short out" the electric field component along the magnetic field. Therefore the concept of moving and "frozen-in" magnetic field lines has been extensively used in space plasma physics.

If there exists regions where the parallel electric field is nonzero, the magnetic field lines can be "cut" so that different segments of a field line move differently, as illustrated by Alfvén and Fälthammar,[8] because the condition in Eq. (4) is violated.

The assumption of vanishing parallel electric fields was challenged by Alfvén[4] who proposed that such fields exist above the ionosphere and cause downward acceleration of auroral primary electrons, but his suggestion was generally disregarded. The first indication in support of Alfvén's idea was McIlwains[6] observation of auroral primary electrons. Since then, an extensive body of evidence from space measurements has been accumulated, and the existence and importance of magnetic-field aligned electric fields in space plasmas is now generally accepted. For example, in the auroral acceleration region upward directed parallel electric fields accelerate electrons down into the ionosphere where they cause aurora; at the same time ionospheric ions go into the magnetosphere. Beautiful examples of such electron and ion fluxes have been reported in Ref. 9. In the auroral return current region, downward directed parallel electric fields accelerate ions down and electrons up.

The question of how it is possible for an essentially collisionless plasma to sustain a parallel electric field has not been fully answered, but a number of possible mechanisms have been identified. (For a recent review see Ref. 10.) Nevertheless, the existence of these parallel fields is an empirical fact, which means that the concept of moving, frozen-in magnetic field lines must be applied with caution. Alfvén seems to have been the first to draw attention to frozen-in magnetic field lines and he became convinced by his studies of the aurora that this concept could be dangerously misleading. Especially in his later years he vigorously warned against unjustified use of the concept.[4]

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The expressions

$$\frac{1}{2}m\dot{r}^2 + r^2\dot{\theta}^2$$

in Eqs. (7) and (8) should be replaced by

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2),$$

and the expressions

$$m\ddot{r} - 2r\dot{\theta}^2$$

in Eqs. (10) and (11) by

$$m(\ddot{r} - r\dot{\theta}^2).$$

The conclusions of the Comment are unchanged.

