Energy around Coils and Magnets

Introduction

This paper studies the magnetic energy stored in the air space surrounding solenoidal coils and bar magnets. Section 1 deals with coils where, for air-cored coils using Nagaoka's formula, a simple method is given for apportioning the magnetic energy between that stored in the internal volume and that stored externally. Coils having a permeable core are also considered. Section 2 shows how this approach can also be used to establish the energy stored external to a permanent magnet. Section 3 examines the situation involving two co-located coils that are energized from separate power sources. It is shown that, when the coils are energized in time sequence, the energy apportioned between each source depends upon the source impedances. The conditions are established whereby, on application of energy to the second coil, the magnetic field gains a larger quantity of energy taken from the power source of the already energized first coil. Section 4 uses this result to show that, when the field from a permanent magnet is enhanced by the field from a coil wound around it, the external field gains significant energy from within the magnet. By analogy to the two-coil situation, it is argued that this energy comes from the quantum domain. Section 5 discusses an alternative model for the permanent magnet in which this quantum domain connection is more obscure. Arguments are presented as to why this alternative model is incorrect.

1. Energy around a Coil from Nagaoka's Formula

1.1. Air-cored coils

The inductance of a single layer air-cored solenoid is given by Nagaoka's formula

$$L = k\frac{\mu_0 N^2 A}{l} \qquad (1)$$

where k is Nagaoka's dimensionless geometric factor (between 0 and 1) that depends upon the length/diameter ratio, N is the number of turns, l is the solenoid length and A is the area.

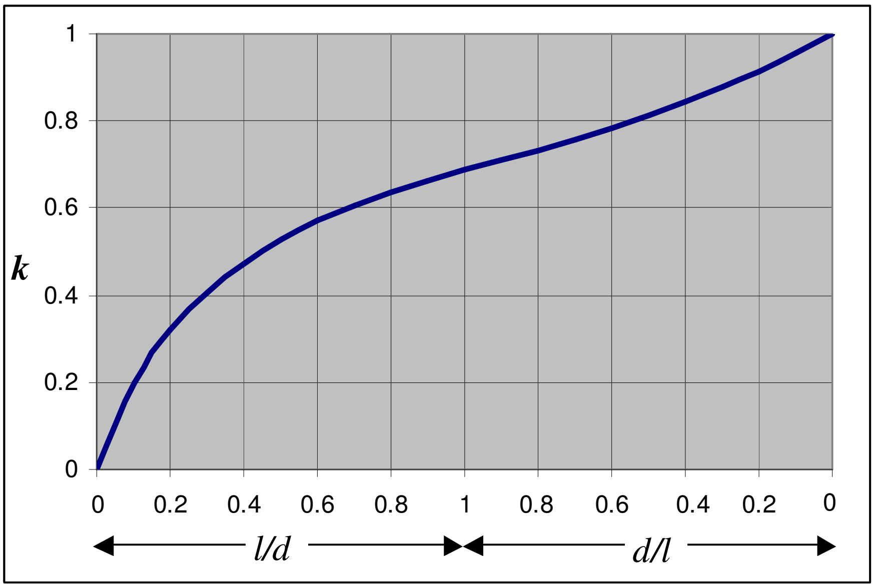


Figure 1. Nagaoka's Geometric factor *k*.

Figure 1 gives k plotted against l/d for lengths l shorter than diameter d, and plotted against d/l for lengths greater than the diameter, thus covering any length up to the infinite solenoid.

Equation (1) can be expressed as

$$L = k\frac{N^2}{R_{INT}} \tag{2}$$

where R_{INT} is seen to be the reluctance of the cylindrical air space enclosed by the coil,

$$R_{INT} = \frac{l}{\mu_0 A} \tag{3}$$

It follows that, when carrying a current i, the total energy W_{TOT} stored in the magnetic field as given by

$$W_{TOT} = \frac{Li^2}{2} \tag{4}$$

can also be expressed as

$$W_{TOT} = k\frac{(Ni)^2}{2R_{INT}} \tag{5}$$

This is the total magnetic energy stored in all space. That total space can be divided into two regions, (a) the cylindrical air space having reluctance R_{INT} enclosed by the coil and (b) the remaining space external to the coil. The flux lines closing through that external space can be considered to flow through a reluctance R_{EXT} as shown in Figure 2.

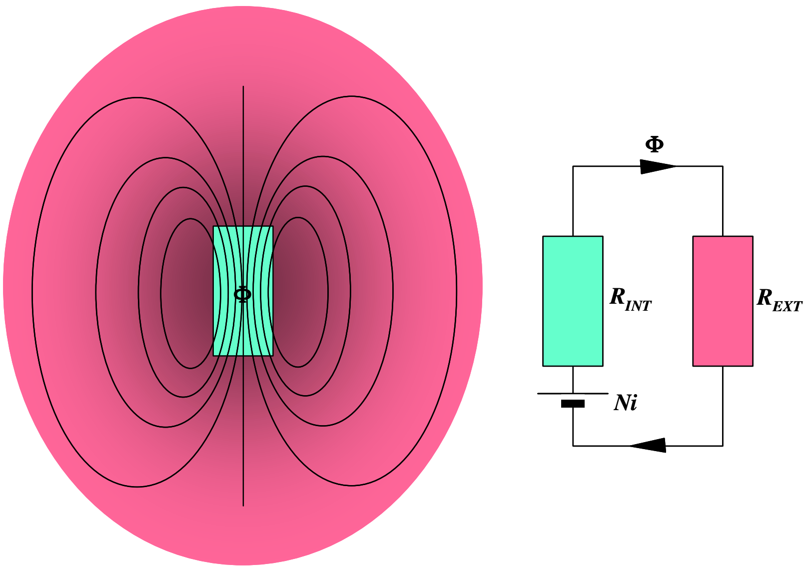


Figure 2. Internal and External Reluctances

To find the energy W_{INT} stored within the cylindrical volume of the coil, we can use the energy formula for a flux Φ within a reluctance R

$$W = \frac{\Phi^2 R}{2} \tag{6}$$

where the actual flux (not flux linkage) is given by $\Phi = \frac{Li}{N}$, yielding

$$W_{INT} = k^2\frac{(Ni)^2}{2R_{INT}} = kW_{TOT} \tag{7}$$

It then follows that the energy W_{EXT} stored in all space *external* to the coil volume is

$$W_{EXT} = (1-k)W_{TOT} = k(1-k)\frac{(Ni)^2}{2R_{INT}} \tag{8}$$

The energized coil can be considered as a mmf generator of Ni ampere-turns driving flux through two reluctances in series representing the internal reluctance R_{INT} and the external reluctance R_{EXT}, see Figure 2. Now since the inductance is given by (2) and also by

$$L = \frac{N^2}{R_{INT} + R_{EXT}} \tag{9}$$

we can equate (2) and (9) to find R_{EXT} as

$$R_{EXT} = \left(\frac{1}{k} - 1\right) R_{INT} \tag{10}$$

Combining (8) and (10) to eliminate R_{INT} yields

$$W_{EXT} = \frac{\Phi^2 R_{EXT}}{2} \tag{11}$$

which gives the external energy in the same form as (6).

1.2. Graphical method

The "Magnetic Ohm's Law" circuit shown in Figure 2 can be solved graphically as shown in the flux v. mmf plotted in Figure 3. The energies stored in the two reluctances, as given by (6), (7) (8) or (11), are then also given by the areas of the two triangles.

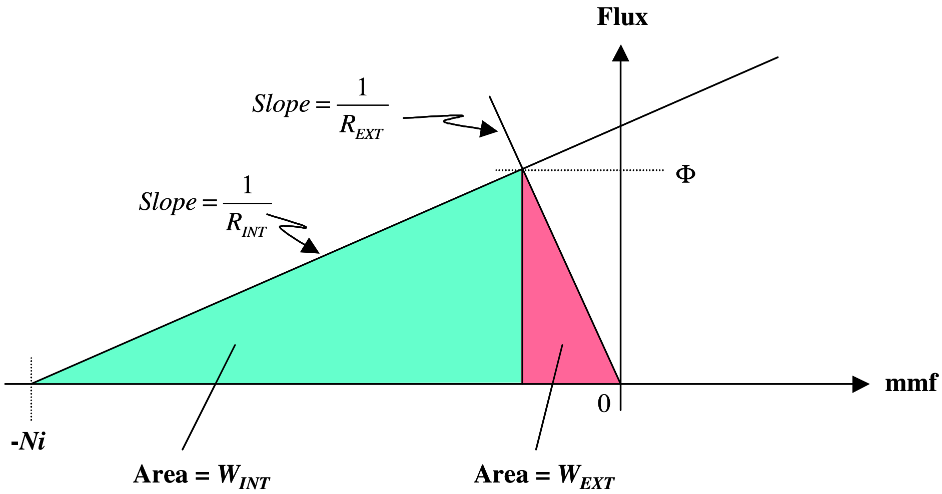


Figure 3. Flux v. mmf for the Solenoid

Showing this chart in the second quadrant with negative Ni may seem odd, it could equally have been shown in the first quadrant using positive Ni. The second quadrant is deliberately chosen to illustrate the similarity to the permanent magnet features in the next section.

1.3. Coils with permeable cores

It is often required to calculate the inductance of a coil wound onto a rod of permeable material, e.g. as in a ferrite rod antenna. Using the reluctance R_{INT} of the *air space* occupied by the ferrite rod we can use (10) to calculate the reluctance R_{EXT} for the external space. The actual reluctance of the ferrite rod is of course $R_{ROD}=R_{INT}/\mu_R$ where μ_R is the relative permeability. Hence using R_{INT}/μ_R in place of R_{INT} in (9) we obtain

$$L_{ROD} = \frac{N^2}{\frac{R_{INT}}{\mu_R} + R_{EXT}} = \frac{N^2}{R_{INT}\left(\frac{1}{\mu_R} + \frac{1}{k} - 1\right)} = \frac{\mu_0 N^2 A}{l\left(\frac{1}{\mu_R} + \frac{1}{k} - 1\right)} \tag{12}$$

For high permeability, which is usually the case for ferrites, and for short rods where $\mu_R >> \frac{k}{1-k}$, the $1/\mu_R$ term becomes negligible and the inductance is then independent of the permeability, being determined solely by the geometry.

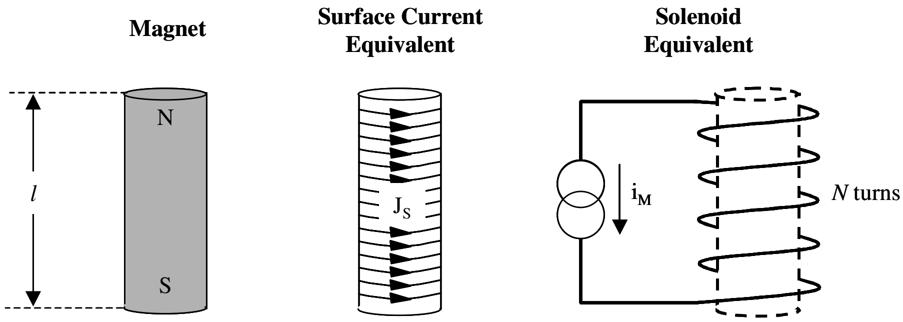
$$L_{ROD} \approx \frac{\mu_0 N^2 A}{l\left(\frac{1}{k}-1\right)} = \frac{N^2}{R_{EXT}} \tag{13}$$

Effectively this says R_{ROD}=0 whence all of the magnetic energy is stored in the external air space; that energy can be determined from (4) using the inductance given by (13) where R_{EXT} is given by (10). For long rods, where the above inequality is no longer true, the *total* energy as given by (4) and (12) can be apportioned between R_{ROD} and R_{EXT} using (6).

2. Energy around a Bar Magnet

2.1. Solenoid Equivalent Model

It is common practice to model permanent magnets by their surface current or solenoid equivalents, see Figure 4. The magnetic material has a uniform distribution of magnetic dipoles yielding a dipole-moment volume-density **M.** This is replaced with an identical volume of *air*, around which flows a surface-current density J_S such that the total dipole moment **M***v* remains the same, where *v* is the volume. Since $M = \frac{B_R}{\mu_0}$ the dipole moment *m* of the magnet is given by $m = \frac{B_R v}{\mu_0}$, where B_R is the remanence. The dipole moment of the surface current model is $m = J_S v$ hence it follows that $J_S = \frac{B_R}{\mu_0}$.



$$J_S = \frac{B_R}{\mu_0} \qquad (Ni)_M = \frac{B_R l}{\mu_0}$$

Figure 4. Bar magnet and Equivalent Models

The effective ampere-turns *(Ni)*$_M$ of the solenoid equivalent model is then given by

$$(Ni)_M = J_S l = \frac{B_R l}{\mu_0} \tag{14}$$

2.2. Energy Considerations

Putting the ampere-turns of (14) in (8) we get

$$W_{EXT} = k(1-k)\left(\frac{(B_R l)^2}{2{\mu_0}^2 R_{INT}}\right) \qquad (15)$$

As before, R_{INT} is the reluctance of the *air space* occupied by the magnet, as given by (3). Those skilled in magnetic systems will recognize this reluctance as something used in the procedure for establishing the load line on the magnet's BH curve. Many people perform this procedure by rote, without appreciating the reason why. As in the solenoid case shown in Figure 2, the unkeepered magnet can be modelled by its "magnetic Ohm's Law" equivalent circuit as a mmf generator driving flux through two reluctances in series representing the internal reluctance R_{INT} and the external reluctance R_{EXT}. Figure 5 shows a graphical method for establishing the flux Φ by applying the R_{EXT} and R_{INT} load-lines to the flux v. mmf chart. Also shown is the graphical procedure for establishing the flux density B from the BH curve that, for magnets where there is no significant domain reversals, performs the same task. It is left to the reader to satisfy himself that these two methods are equivalent.

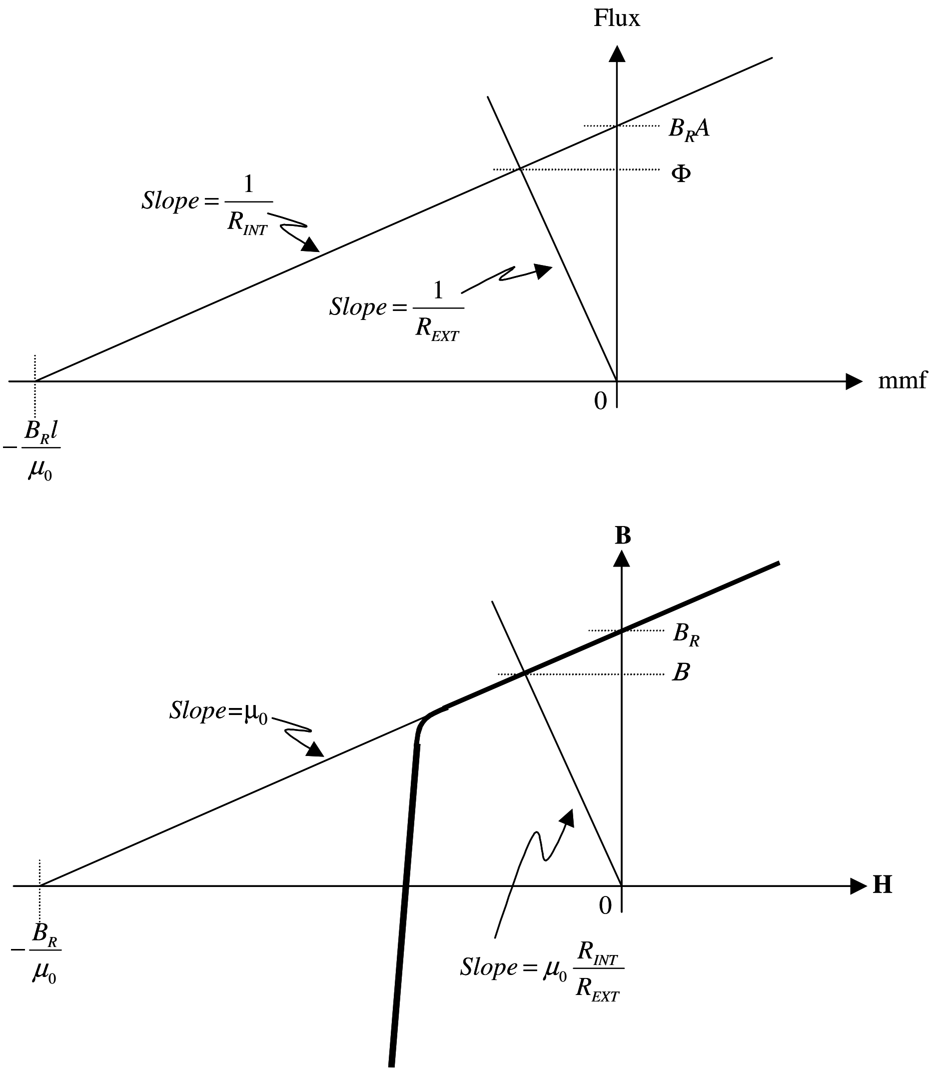


Figure 5. Load-lines for (a) reluctances in series and (b) the BH curve

The point being made here is that the reduction from B_R due to the load-line is normally attributed to a *de-magnetization* factor, whereas there is no actual change of magnetization over that range, M remains constant. The reduction is simply magnetic Ohm's Law applied to a changed value of external reluctance yielding a different "mmf drop" (like voltage drop) across the internal R_{INT}. Nagaoka's geometric k factor, acting like the geometric *de-magnetization* factor, allows calculation for the magnet's operating conditions. In this case the term *de-magnetization* seems inappropriate.

When there *is* change of magnetization, e.g. for a non-linear Alnico BH curve, the graphical method is needed since the reduction in B is then due to two different features.

That a magnet's operating point can be found by applying its mmf across the two reluctances in series brings into focus the presence of that mmf, it is not a math artefact. Hence the negative H value of $\frac{B_R}{\mu_0}$ in the BH curve in Figure 3 is real and represents the distributed ampere-turns of all the atomic current circulations responsible for the magnetization. The only reason that it is negative is because of our choice of where the zero for the BH curve lies, we choose to ignore that atomic contribution (we could have chosen the $-\frac{B_R}{\mu_0}$ point as zero and shown all the H as positive).

Perhaps the arbitrary nature of the H values being negative is best illustrated by using (3) to eliminate R_{INT} from (15), obtaining the external energy as

$$W_{EXT} = k(1-k)\left(\frac{B_R^{\,2}}{2\mu_0}v\right) \tag{16}$$

where v is the volume of the magnet. The term in large parentheses will be recognized as the energy within the *air volume* of the magnet if the internal field is at the B_R level (as it would be if the magnet were keepered). Nagaoka's k factor enables us to calculate the external energy of the unkeepered magnet as a fraction of this value. We can also calculate the internal energy as another fraction of this value

$$W_{INT} = k^2\left(\frac{B_R^{\,2}}{2\mu_0}v\right) \tag{17}$$

Treating the energy within the magnet as being within air eliminates the nonsense that arises from the usual BH curve where its internal H is considered to be of opposite direction to the B, and in particular the perception that when H=0 the internal energy is zero.

2.3. Energy Available to do Work

It is often of interest to be able to calculate the total energy available for a magnet to do work (such as lift a weight) when it attracts a keeper towards itself. At first sight (15) or (16) might be considered for this application, but this gives a low estimate. A full calculation requires math integration as R_{EXT} reduces from the value determined from Nagaoka, when the keeper is at infinity, down to zero when the keeper "shorts out" the magnet. The final result is

$$W_{LIFT} = (1-k)\left(\frac{B_R^{\,2}}{2\mu_0}v\right) \tag{18}$$

3. Energy supplied by two coils

In this section we consider magnetic energy as supplied by two coils each driven from separate power supplies. We use bifilar wound coils so that they are identical, having equal inductance values and supplying magnetic energy to the same internal and external reluctances. The coupling between the two coils is unity. Clearly if both coils are simultaneously energized with identical L/R time constants, the total magnetic energy achieved in space is supplied equally by the two power sources. However if the coils are energized in time sequence the perhaps surprising result is obtained where the first one to be energized supplies more energy. Put simply, transformer action when the *second* coil is energized places an additional voltage transient onto the current already flowing in the *first* coil, hence extracting additional energy from its power source.

Consider a solenoid to be wound with two identical coils, closely coupled, e.g. bifilar wound. Let coil 1 be energized at time t_1 from a high impedance current source, such as a very high voltage V_1 in series with a high resistance R_1. Its inductance gets charged with current rising exponentially with time constant $\frac{L}{R_1}$ to reach a maximum value of $I_1 = \frac{V_1}{R_1}$, storing magnetic energy $W_1 = \frac{LI_1^2}{2}$. During the charging period the voltage across the coil falls exponentially from V_1 to zero, and the time-integral of the power pulse formed from the product of the rising current and falling voltage gives that same W_1 value of energy supplied from the V_1 source, see Figure 6.

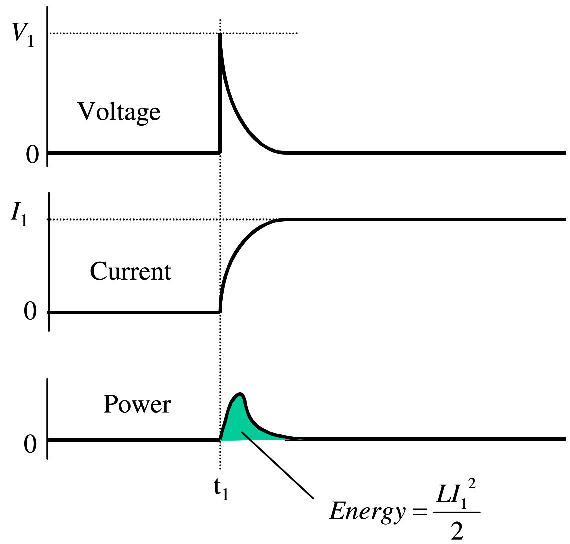


Figure 6. Transients at first coil

Now let coil 2 be energized at a later time t_2 from a low voltage source V_2 having a low internal resistance R_2. Because R_1 is very large, its reflection via transformer action does not influence the charging time constant $\frac{L}{R_2}$ of coil 2, the inductance gets charged with current rising exponentially to reach a maximum value of $I_2 = \frac{V_2}{R_2}$. The V_2 source supplies magnetic energy $W_2 = \frac{LI_2^2}{2}$, Figure 7.

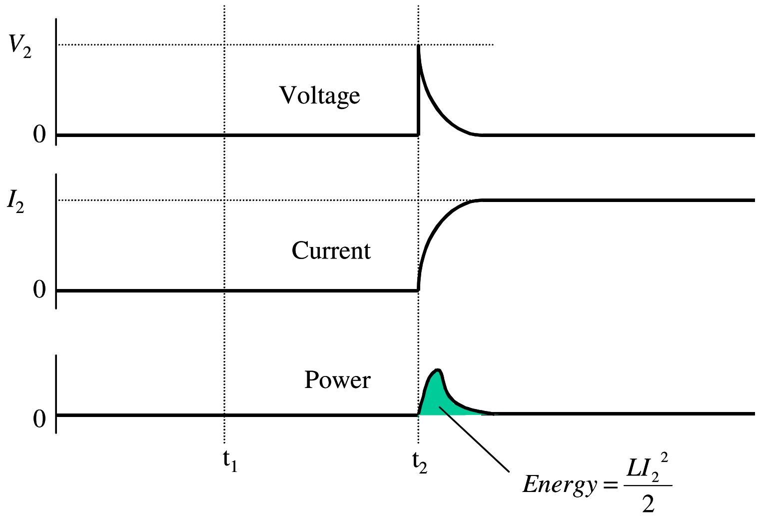


Figure 7. Transient at Second coil

During this second charging phase, by transformer action, coil 1 experiences the same voltage as coil 2, an instantaneous rise to V_2 then falling exponentially to zero, which, multiplied by the current I_1, produces a *second* power pulse drawing energy from source V_1. This second energy pulse from V_1 accounts for the third term $W_3 = LI_1I_2$ which applies to the total magnetic energy from both coils as given by $W_{TOT} = \frac{L(I_1 + I_2)^2}{2}$ where the expansion of $(I_1 + I_2)^2$ yields

$$W_{TOT} = \frac{LI_1^2}{2} + \frac{LI_2^2}{2} + LI_1I_2 = W_1 + W_2 + W_3 . \qquad (19)$$

Figure 8 shows the voltage, current and power waveforms for the W_3 pulse taken from the first coil's power supply.

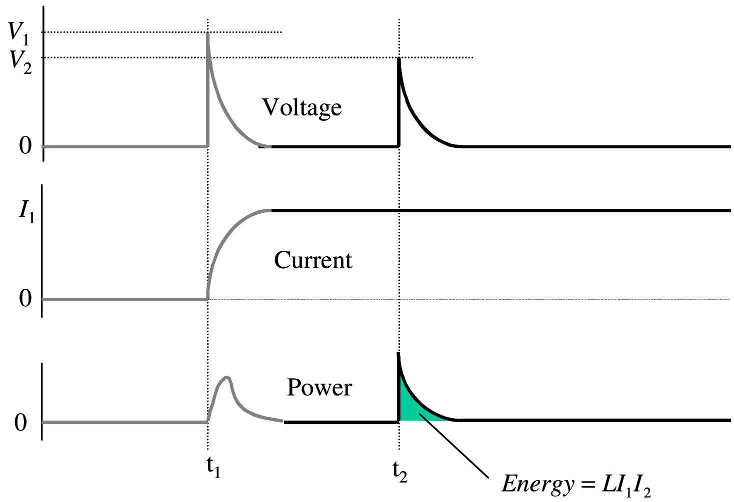


Figure 8. Second Transient at first coil

Clearly, for large values of I_1 compared with I_2, the W_3 term is greater than W_2, thus *the power supply to the first coil to be energized supplies most of the additional magnetic energy*. This is important when considering energy supplied by a magnet plus a coil since the magnet's field already exists.

4. Energy around a Magnet plus Coil

If we now place a coil around the bar magnet, energise that coil with *Ni* ampere-turns, we can evaluate the external energy using the sum value of coil and magnet currents in (8). We obtain

$$W_{EXT} = k(1-k)\left(\frac{((Ni)_M + Ni)^2}{2R_{INT}}\right) \qquad (20)$$

Expanding the squared term and using (14) we find the external energy to be the sum of three values

$$W_{EXT1} = k(1-k)\left(\frac{(B_R l)^2}{2\mu_0{}^2 R}\right) \text{ is the original energy from the magnet} \qquad (21)$$

$$W_{EXT2} = k(1-k)\left(\frac{(Ni)^2}{2R}\right) \text{ is energy supplied by the coil} \qquad (22)$$

$$W_{EXT3} = k(1-k)\left(\frac{(B_R l)(Ni)}{\mu_0 R}\right) \text{ is additional energy } \textit{supplied by the magnet.} \qquad (23)$$

The total energy supplied by the coil (both internal and external energy) is given by (5), hence the ratio of the externally available *additional* energy W_{EXT3} to the supplied coil energy W_{TOT} is

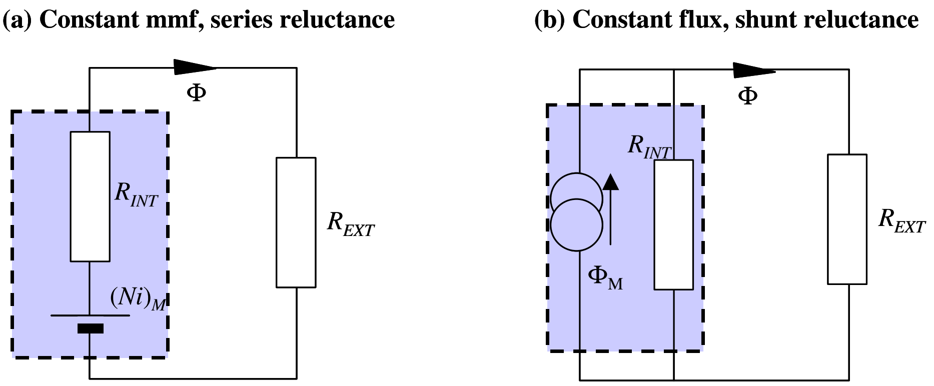
$$\frac{W_{EXT3}}{W_{TOT}} = 2(1-k)\frac{B_R l}{\mu_0 Ni} \qquad (24)$$

If $\frac{B_R l}{\mu_0} >> \frac{Ni}{2(1-k)}$, which is generally the case because of the large values of effective surface current for magnets, then considerably more energy is transposed into space than that supplied by the coil current, the extra energy coming from the magnet.

5. Alternative Magnet Model.

In any circuit, by Thevenin's Superposition theorem, any constant voltage generator V of internal series resistance R can be replaced with an "equivalent" constant current generator I of internal shunt resistance R, where I=V/R. This produces identical external voltage and current levels, but it should be noted that when internal power dissipation is considered the two circuits are not generally equivalent. Only in the special case where the external load equals the internal resistance do the two versions dissipate identical power.

Thevenin's theorem can be used to create an alternative model for a permanent magnet that yields identical external performance, but except for the special case where the external load equals the internal reluctance, with different internal energy considerations. The constant mmf generator plus series internal reluctance model, as described in section 2.1. is replaced with a constant flux generator in parallel with that reluctance. This alternative model is preferred by some scientists since it offers a different energy perspective when considering an energized coil around the magnet. Figure 9 shows the two alternatives .



$$(Ni)_M = \frac{B_R l}{\mu_0} \qquad \Phi_M = \frac{B_R l}{\mu_0 R_{INT}}$$

Figure 9. Alternative Models for a Permanent Magnet

If we now consider the case where additional mmf is supplied from an energized coil wound around the magnet, we obtain the two circuits shown in Figure 10.

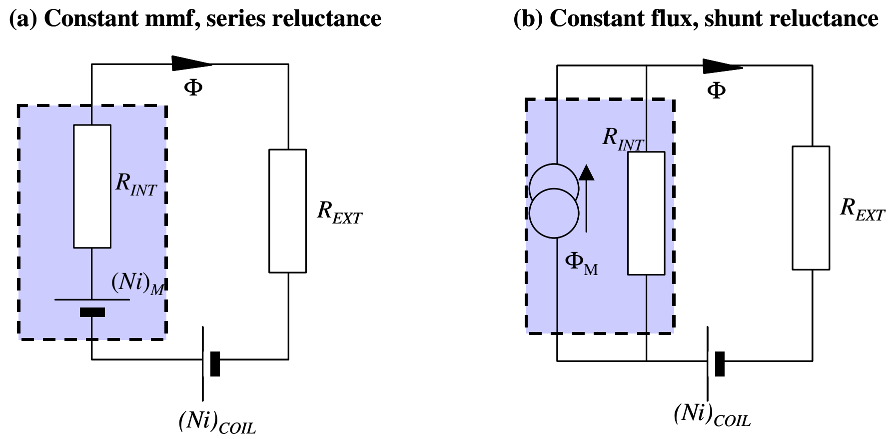


Figure 10. Models with Coil mmf

Whereas with the constant mmf model Fig.10(a) the extra external energy given by (23) comes from the magnet's source mmf *(Ni)*$_M$, using the constant flux model Fig.10(b) the same value of extra energy is seen to come from that previously stored in R_{INT}. This seems to eliminate any connection with the quantum forces driving the magnet's mmf *(Ni)*$_M$. However what is overlooked in this argument is the mmf drop across R_{INT} that applies a different value of mmf to the magnet's source flux Φ_M. If in the one model *(Ni)*$_M$ is a quantum source of energy, supplying energy according to the flux, then also in the other model Φ_M is also a quantum source supplying energy according to the mmf. It is argued by those same scientists that a super-conducting current loop is an example of a constant flux source, hence the atomic current loops responsible for ferromagnetism should be modelled in that way. Again the mmf is overlooked. In the presence of a changing magnetic field a super-conducting loop does not maintain constant current, it's current hence mmf changes so as to keep the flux constant. Its dipole moment also changes. An array of super-conducting current loops, in the presence of additional magnetic flux, will change its magnetization. There is ample evidence that permanent magnets do not respond in this way. Also if atomic dipoles were of this form then FMR experiments would show entirely different results and the well-known gyro-magnetic ratio would have little significance. For these reasons the constant flux model is discarded.