Do Maxwell's equations need revision? - A methodological note

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We propose a modification of Maxwell's macroscopic fundamental set of equations in vacuum in order to clarify Faraday's law of induction. Using this procedure, the Lorentz force is no longer separate from Maxwell's equations. The Lorentz transformations are shown to be related to the convective derivative, which is introduced in the electrodynamics of moving bodies. The new formulation is in complete agreement with the actual set of Maxwell's equations for bodies at rest, the only novel feature is a new kind of electromotive force. Heinrich Hertz was the first to propose a similar electrodynamic theory of moving bodies, although its interpretation was based on the existence of the so called "aether". Examining the problem of a moving circuit with this procedure, it is shown that a new force of induction should act on a circuit moving through an inhomogeneous vector potential field. This overlooked induction force is related to the Aharanov-Bohm effect, but can also be related to the classical electromagnetic field when an external magnetic field is acting on the system. Technological issues, such as the so called Marinov motor are also addressed.

I. INTRODUCTION

The theory proposed by James Clerk Maxwell successfully unified optics and the electrodynamics of moving bodies. In 1855 he tried to unify Faraday's intuitive field lines description and Sir William Thomson's mathematical analogies to the laws of hydrodynamics, in particular, making use of his 1842 analogy relating heat propagation to electrostatic theory. In 1861 Maxwell proposed a complete set of equations including the displacement current, from which the electromagnetic wave equation could be obtained [1]. Despite its success, however, Maxwell's equations are still the subject of conceptual difficulties and controversy. Notwithstanding its widespread technological applications, it remains particularly painful for scientists and engineers to apply Michael Faraday's law of induction [2, 3]. Faraday himself gave an empirical rule to determine when an induced voltage should be expected in a circuit. The explanation of this rule requires the use of Maxwell's equations if the magnetic field changes with time, but the Lorentz force (considered necessary to define the fields **E** and **B**) if the circuit is displaced [4].

The title of this paper is intended to call attention to some of these conceptual difficulties [5, 6, 7, 8]. In our view the changes proposed in this work do not challenge Maxwell's equations as written for bodies at rest. The novelty of our approach lies in a new form for the expected electromotive force and a more systematic presentation of the Maxwell's fundamental set of equations.

We show that the problem of electromagnetic induction can be more clearly understood using an appropriate procedure. Although it is not usually mentioned in the literature, Heinrich Hertz was the first to propose this procedure and give a systematic treatment of Maxwell's macroscopic equations for the case of moving bodies. This inquiry has its origin in the thought that the theory's current difficulties must be intimately connected to the transport process of physical quantities.

II. ELECTRODYNAMICS OF MOVING BODIES

For a better understanding of Maxwell's fundamental set of equations we must return to its main experimental sources: Faraday's law of induction and Ampère's law. To facilitate the analysis we consider all our sources to be in a vacuum. With his creative mind, Maxwell built his theory on two basic concepts imported from fluid theory: the notions of circulation and flux. Whenever the flux of a well-behaved vector field through a given surface S is calculated,

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Gauss's theorem is applied. For the electric field Maxwell's flux equation is (mks units are used throughout)

$$\int_S (\mathbf{E} \cdot \mathbf{n})dS = \int\int\int_V (\nabla \cdot \mathbf{E})dV = \frac{1}{\varepsilon_0}\int\int\int_V \rho dV. \qquad (1)$$

From this a differential equation can be obtained, which merely states the conservation of charge

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}. \qquad (2)$$

Applying the same divergence theorem to the flux of the magnetic field results in another source equation stating the non-existence of magnetic monopoles):

$$\nabla \cdot \mathbf{B} = 0. \qquad (3)$$

It is usual in textbooks (e.g., Ref [9]) following the argument presented by Maxwell himself [10], to assume that the surface S and its enclosed volume V are fixed to a stationary frame of reference $\mathbb{R}$. This is the source of a common misunderstanding in the extension of Maxwell's theory to moving bodies. Indeed, Maxwell never attempted a systematic treatment of electromagnetic phenomena in moving bodies [10, 11].

To obtain the other two differential equations, Maxwell turned to the concept of circulation. From the integral equation we obtain

$$\oint_\gamma (\mathbf{E} \cdot \mathbf{dl}) = -\frac{d\Phi}{dt}, \qquad (4)$$

where the integration is done along a closed curve γ that borders an open surface S. As was the case in the first two equations, γ and S are considered to be at rest in some frame $\mathbb{R}$, Equation 4 duplicates the results of Faraday's experiments. In textbooks, Faraday's law of induction is usually presented in the form of Eq. 4 (e.g., Refs. [9, 12]). Other authors, however, prefer to calculate it through the approach $\mathcal{E} = \int \mathbf{F}.d\mathbf{l}/q$, where $\mathbf{F}$ is the Lorentz force (e.g., Ref. [2] and references therein). By Stoke's theorem, however, we should have

$$\oint_\gamma (\mathbf{E}' \cdot \mathbf{dl}') = \int\int [\nabla \times \mathbf{E}'] \cdot \mathbf{dS}', \qquad (5)$$

where the primes denote a well-behaved vector field as measured by an observer moving along with the circuit, i.e., transported along the the loop γ, with speed v relative to frame $\mathbb{R}$. If the circuit transport is to be considered, the convective derivative operator, D/Dt should be introduced. Here, it is expressed in terms of the divergence and curl operators for greater utility [13]:

$$\frac{D}{Dt} \cdot = \frac{\partial}{\partial t} \cdot + \mathbf{v}(\nabla \cdot) - \nabla \times [\mathbf{v} \times] \cdot \qquad (6)$$

Applying this operator to Eqs. 4 and 5 we obtain

$$[\nabla \times \mathbf{E}'] = -\frac{D\mathbf{B}}{Dt} = -\frac{\partial \mathbf{B}}{\partial t} - \nabla \times [\mathbf{B} \times \mathbf{v}], \qquad (7)$$

where $\mathbf{E}'$ is the electric field measured by an instrument attached to the transported frame $\mathbb{R}'$ and $\mathbf{B}$ is the magnetic field measured in the rest frame. The analysis of bodies at rest leads us to conclude that time variation in the magnetic field at a given point are dependent on the local electric field distribution. Whenever the body is in motion, however, Heinrich Hertz [11] explicitly argues the necessity of following the field along the line of motion and offers a procedure to carry out this calculation [14]. Hertz, the discoverer of electric waves, suggested this method to account correctly for induction phenomena in moving conductors. Unfortunately, his ideas were not recognized because his theory did not agree the results of experiments involving displacement of nonconductors [15].

We can return to the frame $\mathbb{R}$ by transforming the new electric field according to the rule

$$\mathbf{E}' = \mathbf{E} + [\mathbf{v} \times \mathbf{B}], \qquad (8)$$

which is valid in the non-relativistic limit. In an inertial frame at rest with respect to two identical charges placed a distance r apart, each other exerts a force given by $F = q^2/(4\pi\varepsilon_0 r^2)$ on the other. This relation serves to *define* the charge q (an axiomatic presentation of Maxwell's equations is undertaken in Ref. [16]) and is the basis of electrostatics Coulomb's law when written in the $\mathbb{R}'$ inertial frame still has the form

$$\mathbf{F}' = q\mathbf{E}'. \qquad (9)$$

Now we can return to the $\mathbb{R}$ frame (in practice, the laboratory frame) in the non-relativistic limit by using Eq. 8. The well-known expression for the Lorentz force results:

$$\mathbf{F} = q\mathbf{E}' = q\mathbf{E} + q[\mathbf{v} \times \mathbf{B}]. \tag{10}$$

The Lorentz force is therefore a necessary tool whenever Maxwell's equations are written for bodies (circuits) at rest. If physical bodies are not present, the Lorentz force does not need to be treated as an intrinsic part of Maxwell's equations for a macroscopic electromagnetic field. In fact, the Lorentz force is a by-product of Maxwell's equations and the change of variables required to guarantee their covariance. The usual formulation presented in textbooks (e.g., Ref. [9]) fixes the circuit, E and B to the same reference frame. This is an inconsistent approach to Faraday's law, since induction from motion can not be obtained from Maxwell equations.

This relationship (originally proposed by Hertz) nullifies the widespread argument that when used with the Lorentz force law, Maxwell's equations describe all electromagnetic phenomena (e.g., [17] and references therein).

As we shall gradually come to realize, it is more proper to rewrite the general form of Maxwell's fourth equations in a given reference frame as

$$[\nabla \times \mathbf{B}'] = \frac{1}{c^2}\frac{D\mathbf{E}}{Dt} + \mu_0 \mathbf{J}, \tag{11}$$

in terms of the convective current $\mathbf{J}$. In analogy with Equation 8, $\mathbf{B}'$ is the magnetic field measured in the moving frame and $\mathbf{E}$ is the electric field defined in the rest frame. Expanding the convective derivative we obtain

$$c^2[\nabla \times \mathbf{B}'] = \frac{\partial \mathbf{E}}{\partial t} + \frac{\rho \mathbf{v}}{\varepsilon_0} - \nabla \times [\mathbf{v} \times \mathbf{E}] + \frac{\mathbf{J}}{\varepsilon_0}. \tag{12}$$

The first term on the right-hand side is the displacement current [18, 19], and $\mathbf{J}$ is defined as $\rho\overline{\mathbf{u}}$ where $\overline{u}$ is the average velocity of current carriers (conduction electrons) drifting along the conductor lattice (their typical speed in gold, for example, is about 1 mm/s). One of the benefits of writing Maxwell's equation in this form is that there is no longer any need to use the Lorentz equation to understand the electrodynamics of moving bodies [9, 10, 20]. Another advantage is that we recover a higher level of logical consistency in the equations. The induced electromotive force (emf) is given by

$$\mathcal{E} = \oint (\mathbf{E}' \cdot d\mathbf{l}') = \int\int_{S'} [\nabla \times \mathbf{E}'] \cdot d\mathbf{S}'. \tag{13}$$

Then, simply by using Eq. 8 we have

$$\mathcal{E} = -\int\int \left(\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}'\right) - \int\int_{S'} \nabla \times [\mathbf{B} \times \mathbf{v}] \cdot d\mathbf{S}' \tag{14}$$

and consequently

$$\mathcal{E} = -\int\int_{S'} \left(\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}'\right) + \oint_{\gamma} [\mathbf{v} \times \mathbf{B}] \cdot d\mathbf{l}'. \tag{15}$$

The induced electromotive force is given as a sum of the "transformer emf" $\mathcal{E}_{flux}$ obtained by integrating $\partial \mathbf{B}/\partial t$ over the instantaneous area S' (providing emf even in vacuum, as in Kersts's betatron) [21], plus the "motional emf" $\mathcal{E}_{motion}$ obtained with the $[\mathbf{v} \times \mathbf{B}]$ (a material medium must move through a B-field to produce an emf). Maxwell's equations thus become complete in themselves. In order to reconstruct Faraday's induction law, a similar suggestion was also advanced by Rosen and Schieber [22] and Scalon et al [23].

If we redefine the new field as

$$\mathbf{B}' = \mathbf{B} - [\mathbf{v} \times \mathbf{E}], \tag{16}$$

then Maxwell's equation (12) can be written as

$$c^2[\nabla \times \mathbf{B}] = \frac{\partial \mathbf{E}}{\partial t} + \frac{\rho(\mathbf{v} + \overline{\mathbf{u}})}{\varepsilon_0}. \tag{17}$$

Notice that the first term on the right-hand side is the derivative of the local electric field with respect to time. We can now see why it is imperative to introduce a convective derivative (which cannot be reduced to the temporal derivative at a fixed point of space in an inertial frame) into Maxwell's fundamental equations. Hertz and Lorentz interpreted v as the velocity relative to the ether, while according to Emile Cohn, a leading physicist at the University of Strasbourg at the end of XIX century, v should be interpreted as the velocity of matter relative to the fixed stars. Even Cohn referred to it as an absolute velocity (see Ref. [24] and references therein for an historical account).

III.   THE CONVECTIVE DERIVATIVE AND THE LORENTZ TRANSFORMATION

To round out the picture described above, it must be admitted that electromagnetism is based on a hybrid foundation; the rigidity of mechanical laws is interwoven with the notion of flux and circulation. This is a good reason to come back to the convective derivative. We would like to note that if we restrain circuit motion to the x-axis, we have

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}. \tag{18}$$

This operator connects the Eulerian (following the moving frame) and Lagrangian (fixed in a stationary frame) points of view. The operator D/Dt gives the variation in a given quantity with respect to time, as measured along the motion. We can denote this same rate by D/Dt' in the new moving frame, with respect to a new time coordinate t'. By assuming the time dilation effect the two derivatives can be related by $D/Dt = D/\gamma Dt'$, where $\gamma = (1-v^2/c^2)^{-1/2}$, and c is the speed of light in vacuum. It is therefore a simple matter to show that Eq. 18 and the assumption of linearity in t lead to the following transformation between $\mathbb{R}$ and $\mathbb{R}'$) [4]:

$$t' = \gamma\left(t - \frac{(\mathbf{v}\cdot\mathbf{r})}{c^2}\right). \tag{19}$$

(Substituting Eq. 19 into Eq. 18 recovers an identity after taking into account time dilation). We therefore *have* to introduce a γ factor so that Eq.19 can verify Eq. 18, as well to comply with optical phenomena such as the Michelson-Morley experiment. Using the convective derivative obliges us to introduce "a suitable change of variables" (as asserted by Lorentz himself in Ref. [20]). The relation $\gamma Dt' = Dt$ implies that a clock attached to a frame moving with constant speed v relative to a stationary frame will be animated with a slower rhythm.

By the same token, we can transform the spatial coordinates from frame $\mathbb{R}$ to frame $\mathbb{R}'$ by applying the convective derivative, and still obtain an identity if we choose the appropriate transformation for motion along the x-axis:

$$x' = \gamma(x - vt). \tag{20}$$

Substituting this transformation into Eq. 18, we have

$$\frac{Dx'}{Dt} = \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)x'. \tag{21}$$

Eq. 21 gives $Dx'/Dt' = 0$, as expected. Finally, as Eq. 18 leads to Eq. 19 (and complies with the Michelson-Morley experiment), we can see in this procedure a new derivation of the Lorentz's transformations; the spatial coordinate transformation is just a consequence of the assumed symmetry between the inertial frames.

Historically, Voldemar Voigt (in 1887) and Sir Joseph Larmor (in 1900) [25, 26] were the first to derive precursors to the Lorentz transformations (see also [27, 28]). As the above lines suggest, however, time as it appears in the Lorentz transformations is a more appropriate variable than any absolute, "real" variable and is easy to relate to everyday life experience. This has motivated some proposals to change to a more "convenient" time variable in discussing and understanding physical events [29, 30], and led others to discuss the twin paradox in the context of an absolute time, based on cosmological principles [31]. To illustrate the procedure further, we can relate the charge density in a "stationary" frame to that in a "moving" frame. Based on the principle of covariance, we can state that in an inertial frame $\mathcal{R}'$:

$$\nabla\cdot\mathbf{E}' = \frac{\rho'}{\varepsilon_0}. \tag{22}$$

By substituting Eq. 8 into Eq. 22 above, we obtain

$$\nabla\cdot\mathbf{E} = \frac{\rho'}{\varepsilon_0} + [\nabla\times\mathbf{B}]\cdot\mathbf{v}. \tag{23}$$

To simplify the problem, we neglect the displacement current and only consider circuits or particles in translational motion. Hence, when inserting Eq. 11 into the above equation we get

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho' + \frac{(\mathbf{v}\cdot\mathbf{J})}{c^2}\right). \tag{24}$$
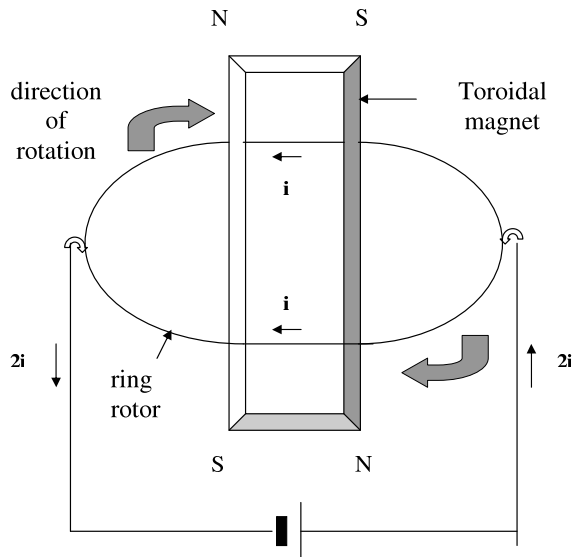
FIG. 1: Schematic of a Marinov motor.

In order to verify the covariance principle we have to modify this equation as follows:

$$\rho' = \left(\rho - \frac{(\mathbf{v} \cdot \mathbf{J})}{c^2} \right). \tag{25}$$

As shown, the above equation is valid in the non-relativistic limit. Taking into account appropriate scaling factors in the space and time coordinates (see, for example, the procedure proposed in Ref. [22]), the γ factor finally appears in Eq. 25.

IV. THE NULL-FIELD FORCE OF INDUCTION

To fully explore the consequences of applying the procedure described above, we must point out that electromotive forces act on a circuit both in the presence of time-varying magnetic fields and through motion of the circuit through an inhomogeneous field according to

$$\mathbf{E}' = -\nabla\phi - \frac{D}{Dt}\mathbf{A} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} - (\mathbf{v} \cdot \nabla)\mathbf{A}. \tag{26}$$

This was already stated by Maxwell (see, for example, Ref. [32]). Supposing that v is constant, we can use vector identities to rearrange this into a more easily interpreted expression:

$$\mathbf{E}' = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + [\mathbf{v} \times \mathbf{B}] - \nabla(\mathbf{v} \cdot \mathbf{A}). \tag{27}$$

The gradient of a scalar field ϕ is a conservative vector field, so the resulting electromotive force receives contributions from three terms. Aside from the well-known forces of induction due to a time-varying, local magnetic field and the motional emf (the second and third terms of Eq. 27), there exists a third force of induction given by

$$\mathbf{F} = -q\nabla(\mathbf{v} \cdot \mathbf{A}) \tag{28}$$

This term, which we might call the null field force of induction, demonstrates the interesting possibility of generating a force of induction even in the absence of a magnetic field provided that a position-dependent vector potential A can be produced.

Although still controversial, an induction motor based on the action of null field induction was apparently conceived by S. Marinov [33]. The concept is simple (see Fig. 1): a permanent magnet of toroidal shape (enclosing its own B-field) is placed in a fixed position inside a conducting ring. The ring is supported by bearings and a direct current is provided through sliding contacts on opposite sides, driving the ring in continuous motion.
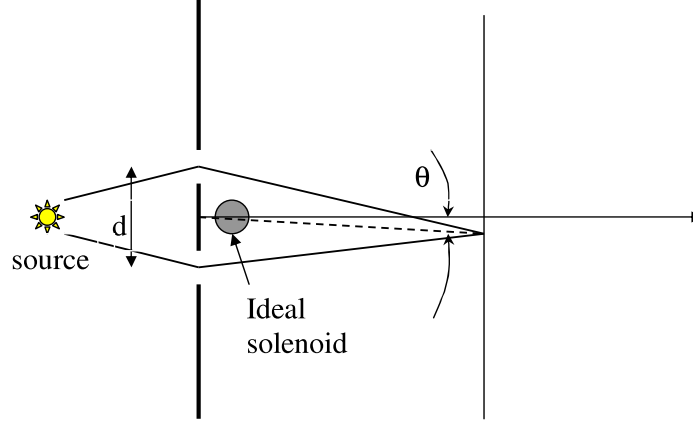
FIG. 2: Scheme of the electron beam deflection on the double-slit experiment. An infinite solenoid is placed between the two slits and creates on the exterior a vector potential $\mathbf{A}$.

An advanced explanation for the forces driving this motor can be based on the this third force of induction [34]. The existence of an applied torque in the Marinov motor apparently points out that the Lorentz force law is not enough to describe all observable electromagnetic forces [35, 36]. This interesting and unusual property is also seen in the Aharonov-Bohm effect [37]. Aharonov and Bohm devised a hypothetical diffraction grating experiment, in which the vector potential would influence not only the interference pattern but also the momentum of the diffracted beam [34, 37] (see Fig. 2). It is worthwhile to note that there exist two different types of effects which are not always clearly distinguished in the literature: i) a deflection of the entire interference pattern due to a classical force; and ii) a deflection of the double-slit interference pattern only [38]. In the two-slit experiment considered by Aharonov and Bohm [37], an infinite solenoid is placed between the two slits so that the magnetic field is zero on all paths. If we consider the action of the null field induction force on the electron beam we obtain

$$F = qE_3 = -q \int \frac{1}{r} \frac{\partial}{\partial \theta} (vA_\theta) r \frac{d\theta}{\Delta t} = i[(vA)_{\theta+\delta\theta} - (vA)_\theta], \tag{29}$$

where E_3 is the null field emf contribution previously referred to, i is the electric current, and θ is the deflection angle of the electrons (see Ref. [38] for a more detailed explanation). We have applied an appropriate change of variables $dq\mathbf{v} = id\mathbf{s}$, where $d\mathbf{s}$ is the circuit line element. It is worthwhile to mention that this term can also generate a classical force if $A = Bx$ (in the presence of a magnetic field), involving a deflection of the electron beam. Without an external field ($\mathbf{B} = 0$), on the other hand, it can generate the Aharonov-Bohm effect which involves no average deflection of the electron beam, but only a deflection of the double-slit pattern [38]. It is clear now that this electromotive force works in open currents, but will it work in closed loops? According to the properties of the gradient operator, we should also expect a null field electromotive force in loop currents. Unfortunately, there is still no clear experimental evidence for this motive force in closed loops. Nevertheless, Trammel's [39] discussion of the Aharonov-Bohm paradox and referral to the peculiar quantum significance of the vector potential is not quite true. Even in the classical limit, this same paradoxical behavior occurs. This point is also clearly discussed in Ref. [38]. In particular, Trammel shows that the forces between a charged particle q, $\mathbf{F}_q$, and a current-carrying body, $\mathbf{F}_j$, are not equal and opposite but obey the relation (see also note [40] and Ref. [41])

$$\mathbf{F}_q + \mathbf{F}_j = -\frac{D}{Dt} q\mathbf{A}. \tag{30}$$

As the interaction between charges and loops must be counterbalanced by mechanical means, the corresponding mechanical force should be

$$\mathbf{F}_{ME} = -\mathbf{F_q} - \mathbf{F_j} = \frac{D}{Dt}(q\mathbf{A}). \tag{31}$$

Applying Eq. 30 to determine the force that a charged particle exerts on a structural ring allows one to subsequently obtain the angular velocity of the transmitted rotation [39].

V. FURTHER CONSIDERATIONS

As a matter of fact, there is a group of affine transformations which leave Maxwell's equations invariant provided appropriate definitions of the magnetic and electric fields are adopted [42]. Maxwell's equations, while containing all electrodynamics, are an indeterminate system. Usually, the following expressions are used for the basic fields $\mathbf{E}$ and $\mathbf{B}$ in an isotropic medium (and inertial frame):

$$\mathbf{D} = \varepsilon \mathbf{E}; \ \ \mathbf{B} = \mu \mathbf{H}; \ \ \mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}^{ext}), \tag{32}$$

where the quantities ε and μ characterize the properties of the medium and $\mathbf{E}^{ext}$ denotes an external electric field. The dependencies $\mathbf{D} = \mathbf{D}(\mathbf{E})$ and $\mathbf{H} = \mathbf{H}(\mathbf{B})$ result from the Poynting theorem. Some other more general relation such as $\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B})$ or $\mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B})$, however, is also possible [43] .

These constitutive equations must be listed among Maxwell's fundamental equations. These relations do not, however, ensure that the three-dimensional vectors of the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, will obey the correct relativistic transformations, only the 4-vectors E^α and B^α are well-defined quantities. This question was well treated in Ref. [44]. It is still unclear how one should extend relativistic electrodynamics to material media, and in particular there are recognized difficulties when the medium is in rotation [45].

It has been argued historically that relativity emerged because Maxwell's equations are not invariant under Galilean transformations, but this statement must be treated with due caution [42, 46, 47]. According to Einstein's theory of relativity, the operational meaning of time is essentially that of a system of local clocks synchronized with each other by optical means. Of course, to synchronize these clocks some procedure must be specified: Einstein assumed communication through light, with equal speeds in both directions. Other authors, not satisfied with this state of affairs, have proposed alternative theories [29, 30] in which time and space coordinates on a "moving" frame are *defined* by the "stationary" observer in frame $\mathcal{R}$ through optical means.

We are far from the clear consensus on the local meaning of time envisioned by Eddington: the truth is that the definition of time is a problem of extreme complexity [48].

VI. CONCLUSION

Through an appropriate modification of Maxwell's equations using the convective derivative Faraday's law of induction is clarified, giving new insight into electromagnetic phenomena. The convective derivative operator can serve as a foundation for the Lorentz transformations, and the difficulty of relating the time variable to daily experience in a clear-cut manner is intrinsically related to these issues.

It was shown that a third force of induction, the null-field force, must act on an electric circuit in the presence of a vector potential. This interesting phenomenon is likely to be useful in numerous practical applications.

Acknowledgments

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[14] From recent reading I learned that H. Hertz asserts: "These statements suffice for extending to moving bodies the theory already developed for bodies at rest; they clearly satisfy the conditions which our system of itself requires, and it will be shown that they embrace all the observed facts" (in Ref. [11], p. 244).

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[40] Eq. 30 should be in all rigor written in terms of the tangential A-potential vector $\mathbf{A}^T$ (swirlling component) such as $\mathbf{F_{ME}} = \frac{D}{Dt}(q\mathbf{A}^T)$. According to the Helmholtz theorem any vector field in a simply connected space can be decomposed into the sum of a longitudinal (irrotational, curl-free) field and a transverse (solenoidal, divergence-free) field. The magnetic field is *defined* through the relation $\mathbf{B} \equiv [\nabla \times \mathbf{A}]$. However, it can exists a longitudinal $\mathbf{A}^L$ vector such as, by definition, $[\nabla \times \mathbf{A}^L] = 0$. $\mathbf{A}^L$ is due to the longitudinal advance of the current between successive turns in a coil. $\mathbf{A}^T$ is invariant under gauge transformation.

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