Since $\omega_\circ$, ω_d, and α are directly related to the roots of the characteristic equation, s_1 and s_2 (see Equations 12.60, 12.61, and 12.62), it should also be clear at this point why s_1 and s_2 are called the natural frequencies of the system.

Equation 12.60 indicates that α, ω_d, and $\omega_\circ$ respectively, form the two sides and hypotenuse of a right triangle. In fact, this is part of a more comprehensive picture of the location in the complex plane of the roots of the characteristic equation. These locations, given by Equations 12.61 and 12.62, are shown in Figure 12.17. Note that as R varies so do α and ω_d; $\omega_\circ$, however, remains constant so that s_1 and s_2 remain the constant distance $\omega_\circ$ from the complex-plane origin for the case of under-damped dynamics.

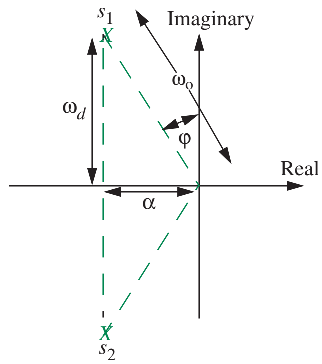

FIGURE 12.17 The location of s_1 and s_2 in the complex plane.

In contrast to the circuit behavior studied in Section 12.1, Equations 12.63 and 12.64 are characterized by two important rates, or frequencies. The first frequency is ω_d, which determines the rate at which the states oscillate. The second frequency is α, which determines the rate at which the states decay. As a consequence, another important characteristic of the circuit behavior described by Equations 12.63 and 12.64 is the relative size of α with respect to $\omega_\circ$. This is usually expressed in terms of the *Quality Factor* Q of the circuit defined by

$$Q \equiv \frac{\omega_\circ}{2\alpha}. \tag{12.65}$$

For the series circuit shown in Figure 12.15, Q is evaluated by substituting Equations 12.44 and 12.45 into Equation 12.65. This yields

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}. \tag{12.66}$$

If the damping factor α is small compared to $\omega_\circ$, as is characteristic of an under-damped circuit, then Q will be large, and the circuit will oscillate for a long time near the frequency $\omega_\circ$. For the series circuit, this is achieved with a small R, that is, with the corresponding resistor near a short circuit. Alternatively, to purposefully damp any oscillations and make them slower, one would make R large.

The preceding discussion suggests an interesting interpretation of Q. From Equations 12.63 and 12.64 we see that the period of oscillation of the circuit states is $2\pi/\omega_d$. Thus, the period of Q oscillations is $2\pi Q/\omega_d$. In the latter period of time the same equations show that the amplitude of the circuit states will decay by

$$e^{-2\pi Q\alpha/\omega_d} \approx e^{-\pi}$$

for $\omega_d \approx \omega_\circ$. Thus, as illustrated in Figure 12.18, the state amplitudes of an under-damped circuit will decay to approximately $e^{-\pi}$, or 4%, of their original values in Q cycles of oscillation.[6]

6. Or to approximately 20% of their original values in $Q/2$ cycles.

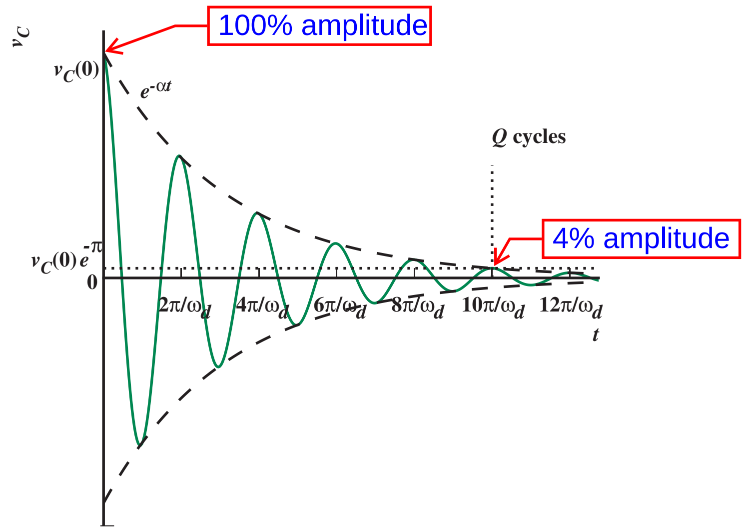

FIGURE 12.18 Waveform of v_C in under-damped, undriven, series RLC circuit for the case of $i_L(0) = 0$, with a Q of 5.

12.2.2 OVER-DAMPED DYNAMICS

The case of over-damped dynamics is characterized by

$$\alpha > \omega_\circ$$

or, after substitution of Equations 12.44 and 12.45, by

$$R/2 > \sqrt{L/C}.$$

In this case, the quantity inside the radicals in Equations 12.46 and 12.47 is positive, and so both s_1 and s_2 are real. For this reason, the dynamic behavior of v_C and i_L, as expressed by Equations 12.58 and 12.59, does not exhibit oscillation. Rather, it involves two real exponential functions that decay at different rates, as the two equations show.

The expressions for v_C and i_L for the case of $i_L(0) = 0$ with over-damping are obtained from Equations 12.58 and 12.59, and are shown here:

$$v_C(t) = \frac{s_2 v_C(0)}{(s_2 - s_1)} e^{s_1 t} + \frac{s_1 v_C(0)}{(s_1 - s_2)} e^{s_2 t} \tag{12.67}$$

$$i_L(t) = -s_1 \frac{C s_2 v_C(0)}{(s_2 - s_1)} e^{s_1 t} - s_2 \frac{C s_1 v_C(0)}{(s_1 - s_2)} e^{s_2 t}. \tag{12.68}$$

Since $\alpha > \omega_\circ$ for over-damped circuits, note that s_1 and s_2 are both real in the preceding two equations.

Figure 12.19 compares the waveforms of v_C and i_L for the case of $i_L(0) = 0$ with under-, over-, and critical-damping. We will address the critically-damped