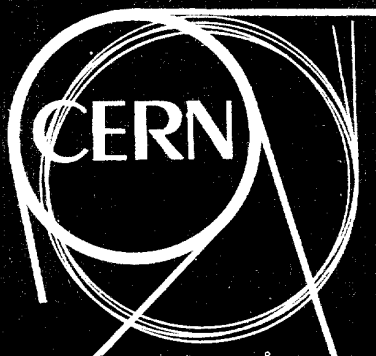




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## Cours/Lecture Series

1992 - 1993 ACADEMIC TRAINING PROGRAMME

### LECTURE SERIES

SPEAKER : Michael BERRY / University of Bristol  
TITLE : Geometric Phases  
TIME : 8, 9, 10 February, 11.00 to 12.00 hrs  
PLACE : Auditorium



Acad Train  
276

### ABSTRACT

*An elementary account will be given of the mathematical phenomenon of "global change without local change", i.e. anholonomy, applied to phases in quantum mechanics and angles in classical mechanics. Phases and angles in photons, electrons, and molecules will be discussed, with a historical emphasis. More advanced topics may include a detailed illustration of the vector and scalar gauge forces associated with the geometric reaction of a light system or a heavy one (modern Born-Oppenheimer theory) and high-order adiabatic corrections to the geometric phase and its relation to recent developments in asymptotics.*

# **Some geometric phases**

transparencies for three lectures at CERN

February 1993

**Michael Berry**

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## Some references

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# Lecture 1

(Some geometric phases, Michael Berry Geneva 1993)

# SOME GEOMETRIC PHASES

1.1

Three central ideas

## 1) Adiabaticity

Mechanical phenomena on the border between statics and dynamics

study of things, i.e. states that do not change; in quantum speak: eigenstates of Hamiltonian operators

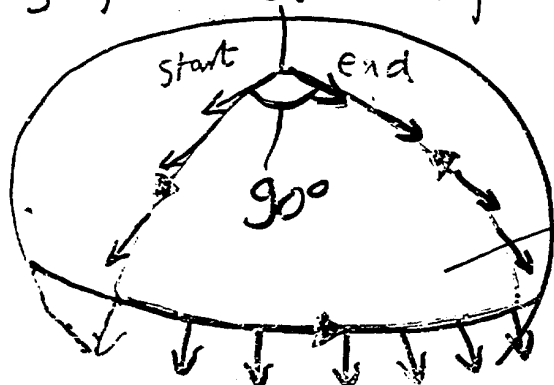
study of happenings: evolving states, changing external fields

Border (the adiabatic regime): study of slow changes

## 2. Anholonomy global change without local change

In geometry: when some quantities are varied round a cycle, others, dependent on them\*, fail to return. e.g. parallel transport of vectors on a sphere

\*and have no local rate of change

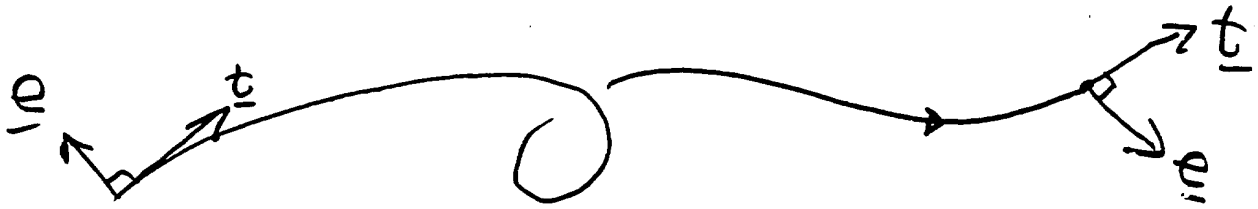


solid angle  
 $\Omega = \frac{1}{8} \text{ sphere} = \frac{1}{8} 4\pi$   
 $= \pi/2$

# Anholonomy in optics Vladimirovsky 1941

1.2

Light rays curved in space :



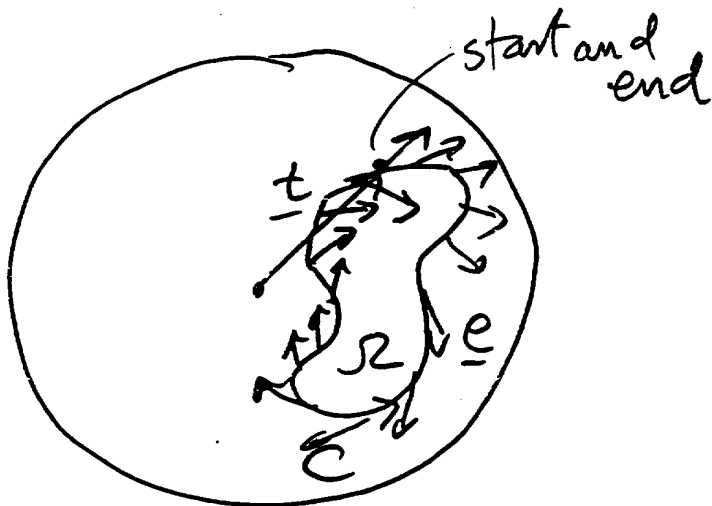
Curving of  $\underline{t}$  governed by Snell's law (1611)

What about polarisation  $\underline{e}$  ?

This is parallel-transported : no rotation of  $\underline{e}$  about  $\underline{t}$ . So  $\underline{e}$  need not return

with  $\underline{t}$  : it rotates by the solid angle  $\Omega$

that the  $\underline{t}$  circuit subtends on its sphere :



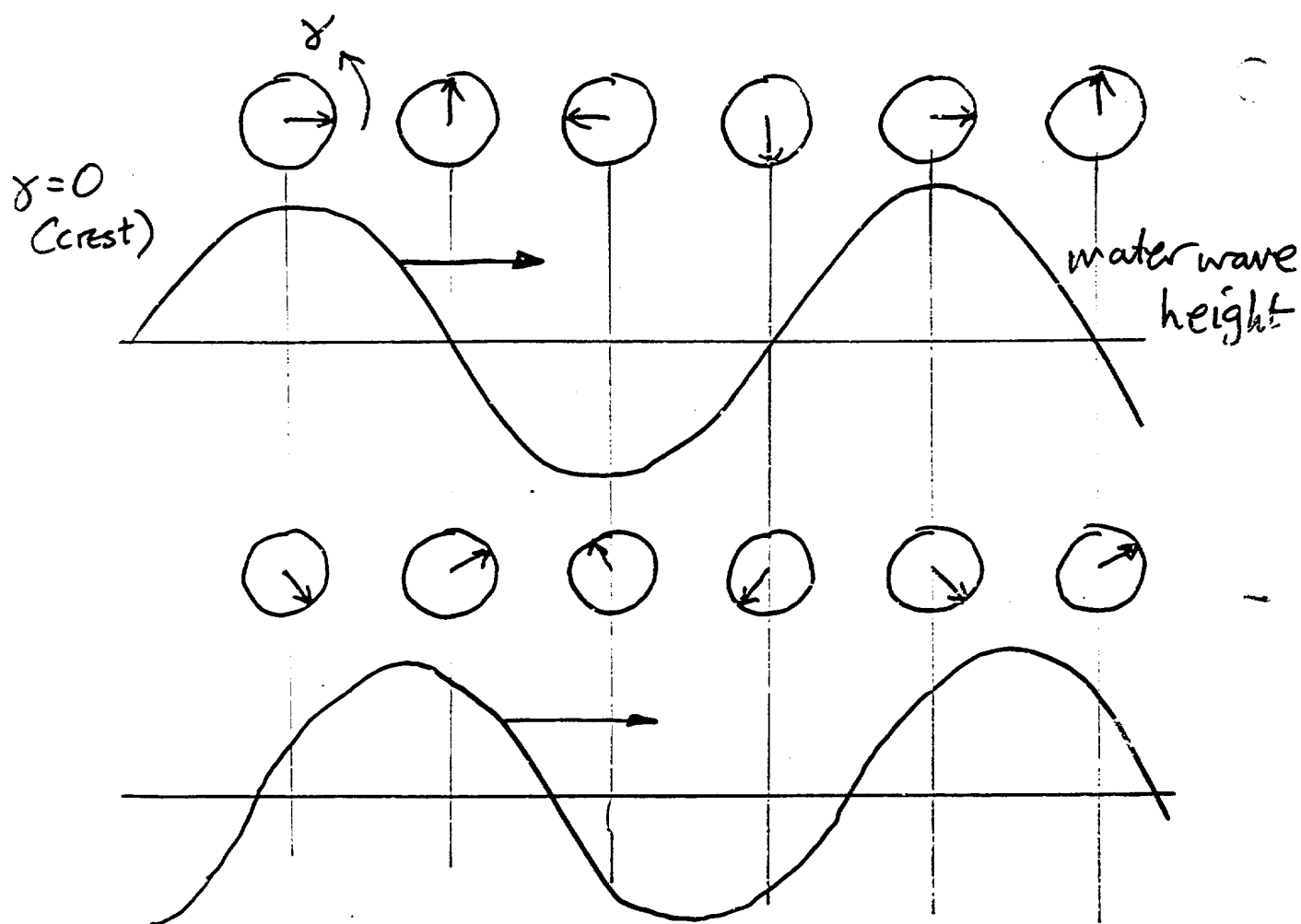
### 3. Phase

1.3

Oxford English Dictionary:

"A particular stage or point in a recurring sequence of movements or changes, e.g. a vibration or undulation, considered in relation to a reference position or time" (1861)

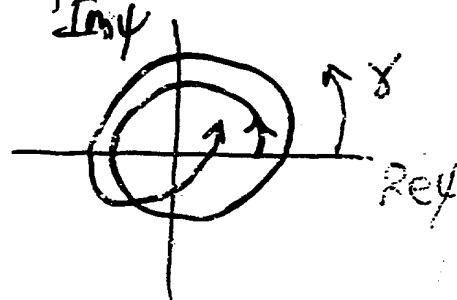
Naturally measured by an angle,  $\gamma$ .



Representation in terms of complex functions of time:

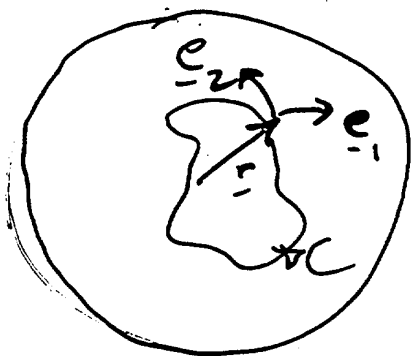
$$\psi(t) = p(t) \cos \gamma(t) + ip(t) \sin \gamma(t)$$

$$= p(t) e^{i\gamma(t)}$$



# Solid-angle formula for parallel transport <sup>1.4</sup>

Triad  $\underline{r}, \underline{e}_1, \underline{e}_2$   $\underline{r}(t)$  radial, driving  $\underline{e}_1, \underline{e}_2$



Let  $\underline{e}$  be any vector  $\perp \underline{r}$

(e.g.  $\underline{e}_1$  or  $\underline{e}_2$ ). Then

$$\dot{\underline{e}} = \underline{\Omega} \wedge \underline{e} \quad \underline{\Omega} = \text{angular velocity}$$

Most general

$$\underline{\Omega} = a \underline{r} + b \underline{\dot{r}} + c \underline{r} \wedge \underline{\dot{r}}$$

parallel transport (no twist about  $\underline{r}$ )  $\rightarrow a = 0$

then  $\underline{e} \cdot \underline{r} = 0 \rightarrow b = 0, c = 1$

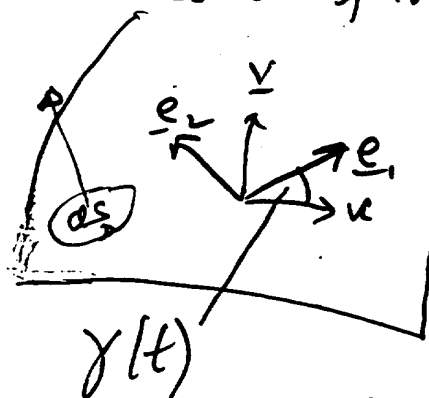
$$\therefore \underline{\Omega} = \underline{r} \wedge \underline{\dot{r}}$$

$$\underline{\dot{e}} = (\underline{r} \wedge \underline{\dot{r}}) \wedge \underline{e} = -\underline{e} \cdot \underline{\dot{r}} \underline{r}$$

Useful expression for equation of motion:

let  $\underline{\phi} \equiv \underline{e}_1 + i \underline{e}_2$ . Then  $\underline{\phi}^* \cdot \underline{\dot{\phi}} = 0$

Chart passage of  $\underline{e}_1, \underline{e}_2$  relative to given orthogonal coordinates on sphere, with unit directions  $\underline{u}, \underline{v}$ , angle  $\gamma$ .



let  $\underline{n} \equiv \underline{u} + i \underline{v}$

then  $\underline{\phi} = \underline{n} e^{-i\gamma}$

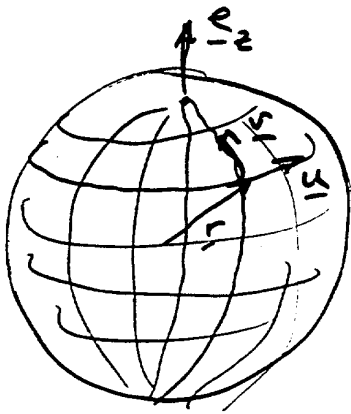
and  $\underline{\phi}^* \cdot \underline{\dot{\phi}} = 0 \rightarrow \dot{\gamma} = \text{Im } \underline{n}^* \cdot \underline{\dot{n}}$

$$\therefore \gamma = \text{Im} \oint \underline{n}^* \cdot \underline{\dot{n}} dt = \text{Im} \iint_{\partial S=C} \underline{\nabla n}^* \cdot \underline{n} \underline{\nabla n} \cdot \underline{d}$$

$\underline{n}, \underline{d}$  on sphere  
= scalar product

$$\gamma(C) \equiv \text{Im} \iint_{\partial S=C} \underline{dn}^* \cdot \underline{n} \underline{dn}$$

For  $\underline{u}, \underline{v}$  choose e.g. polar coordinate directions



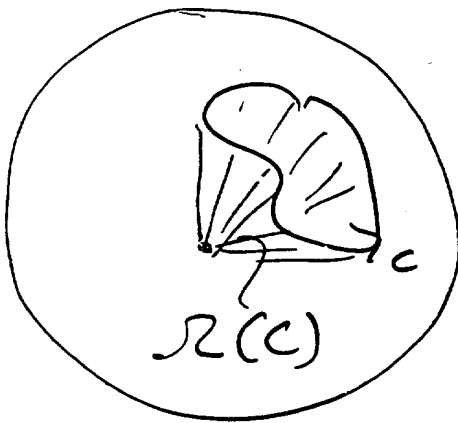
$$\underline{u} = \frac{\underline{r} \wedge \underline{e}_z}{|\underline{r} \wedge \underline{e}_z|}, \quad \underline{v} = \frac{\underline{r} \wedge \underline{u}}{r}$$

$$\text{Then } \boxed{\text{Im } \nabla \underline{n}^* \wedge \nabla \underline{n} = \frac{\underline{r}}{r^3}}$$

= monopole field

$$\therefore \boxed{\gamma(C) = \iint_{\partial S=C} \frac{\underline{r} \cdot d\underline{S}}{r^3} = \Omega(C)}$$

Solid angle subtended by  
C at  $\underline{r}=0$

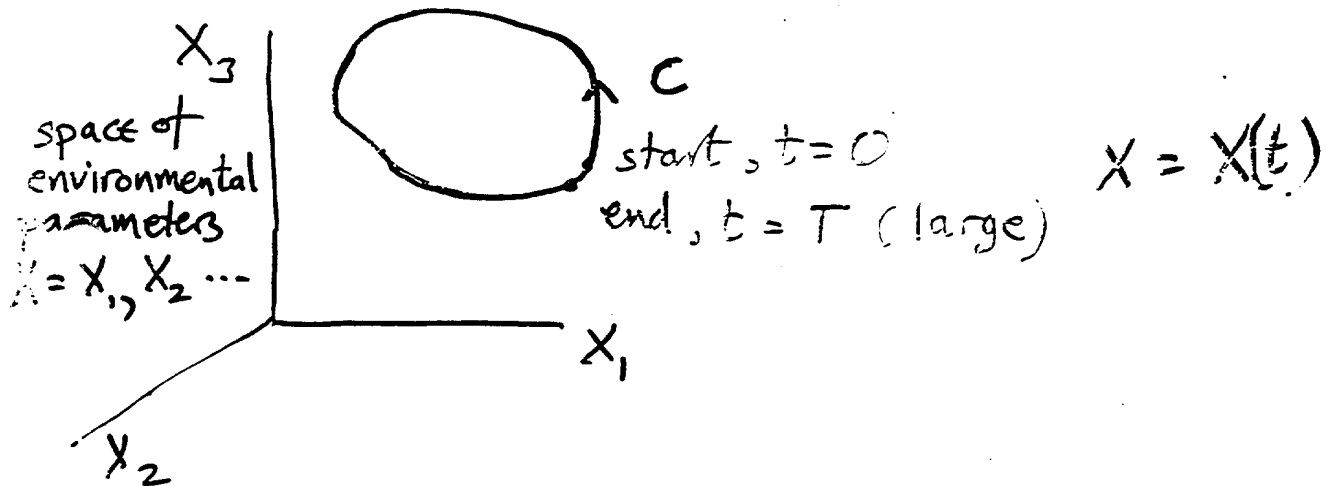


2-form notation: if  $X_1, X_2$  are coordinates on sphere

$$\underline{dn}^* \wedge \underline{dn} \equiv \left( \frac{\partial \underline{n}^*}{\partial X_1} \cdot \frac{\partial \underline{n}}{\partial X_2} - \frac{\partial \underline{n}^*}{\partial X_2} \cdot \frac{\partial \underline{n}}{\partial X_1} \right) dX_1 dX_2$$

## The geometric phase

Take any quantum system (atom, electron, molecule) in one of its discrete energy states (labelled  $n$ ) and slowly alter its environment (e.g. electric or magnetic forces) round a cycle.



Afterwards, environment has returned, and system is still in state  $n$  (quantum adiabatic theorem). But its phase has changed, by  $\gamma = \gamma(T) - \gamma(0)$ .

First guess at  $\gamma$ :

Imagine the environment doesn't change. Then

$$\gamma = -\omega(X)T = -\frac{E_n(X)}{\hbar}T \quad (\text{Planck})$$

energy of state  $n$

When  $X$  changes, generalize to

$$\gamma = (?) = -\frac{1}{\hbar} \int_0^T dt E_n(X(t))$$

But this is wrong. The true phase is ..



$$\gamma = -\frac{1}{\hbar} \int_0^T dt E_n(X(t)) + \gamma_n(C)$$

"dynamical phase"

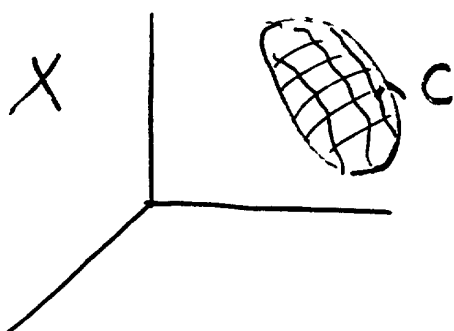
"geometric phase"

dynamical phase: increases with  $T$ ; system's answer to:

"how long did your trip take?"

geometric phase: independent of  $T$ , dependent only on  $n$  and the geometry of  $C$ ; system's answer to:

"where have you been?"



$\gamma_n(C)$  is phase anholonomy

As with other circuit-dependent quantities in physics,

$\gamma_n(C)$  = flux of something through  $C$

(e.g. emf = flux of rate of change of magnetic field)

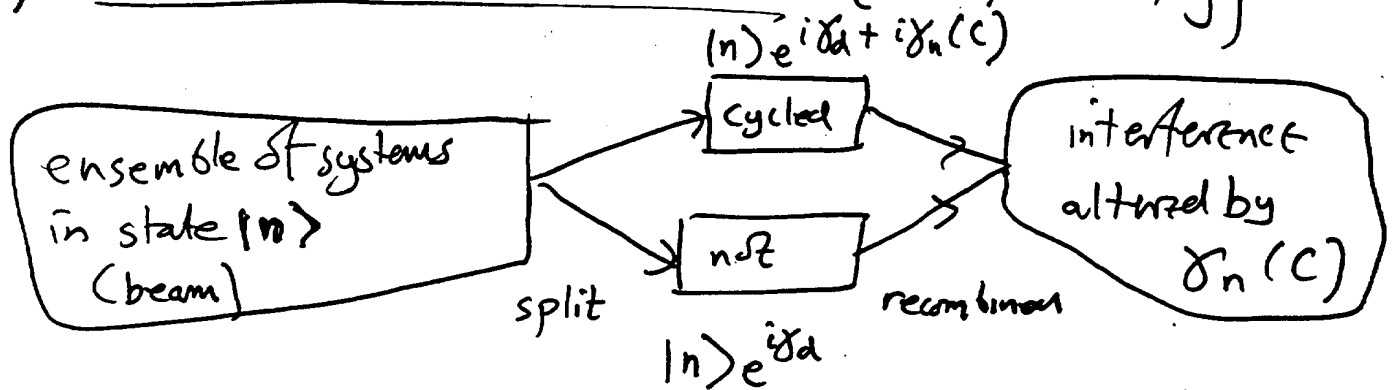
Here (from Schrödinger equation)

$$\gamma_n(C) = - \iint_{\partial S = C} \text{Im} \langle dn | n \rangle dn \quad \text{abstractly (2 form)}$$

$$= - \iint_{\partial S = C} dX_1 dX_2 \left( \iint_{\text{coordinates spanning } C} dr \text{Im} \left[ \frac{\partial \psi_n^*(r; X)}{\partial X_1} \frac{\partial \psi_n(r; X)}{\partial X_2} - \frac{\partial \psi_n^*(r; X)}{\partial X_2} \frac{\partial \psi_n(r; X)}{\partial X_1} \right] \right) \quad \text{concretely}$$

## 3 types of phase experiment

### 1) Two Hamiltonians, one state (interferometry)



### 2) One Hamiltonian, two states (superposition)

$$|\psi(0)\rangle = a_m|m\rangle + a_n|n\rangle$$

$$|\psi(T)\rangle = a_m|m\rangle e^{i\gamma_{dm} + i\gamma_m(c)} + a_n|n\rangle e^{i\gamma_{dn} + i\gamma_n(c)}$$

measure any operator  $\hat{A}$  not commuting with  $\hat{H}$ :

$$\begin{aligned} \langle \psi(T) | \hat{A} | \psi(T) \rangle &= |a_n|^2 \langle n | \hat{A} | n \rangle + |a_m|^2 \langle m | \hat{A} | m \rangle \\ &+ 2 \operatorname{Re} a_n^* a_m \langle n | \hat{A} | m \rangle e^{i(\gamma_{dn} - \gamma_{dm} + \gamma_n(c) - \gamma_m(c))} \end{aligned}$$

depends on  $\gamma_n(c)$

### 3) Repeated cycling

$$|\psi(nT)\rangle = e^{-\frac{i}{\hbar} \int_0^T E_n(t) dt} e^{i n \gamma_n(c)} |\psi(0)\rangle$$

$$= e^{-i(\bar{\omega} - \frac{\gamma_n(c)}{T}) nT} |\psi(0)\rangle \quad \left( \bar{\omega} = \frac{1}{\hbar T} \int_0^T E_n(t) dt \right)$$

$$|\psi(nT)\rangle \equiv e^{-i \omega' nT} |\psi(0)\rangle \rightarrow \text{frequency shift}$$

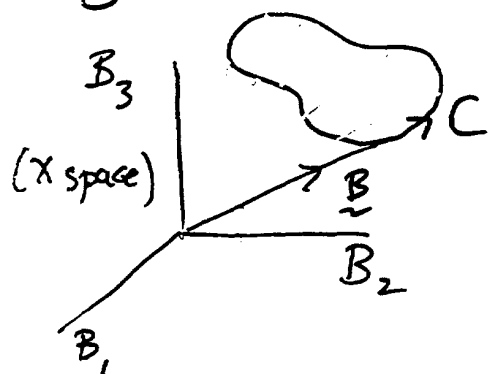
## Example TURNING SPINS

1.9

System: quantum spins

environment: whatever can change spin, e.g.  
a magnetic field  $\underline{B}$  for particles  
with magnetic moment.

cycle: slow circuit in  $\underline{B}$  space



Quantum states labelled by  
component  $n\hbar$  of angular  
momentum along  $\underline{B}$ .

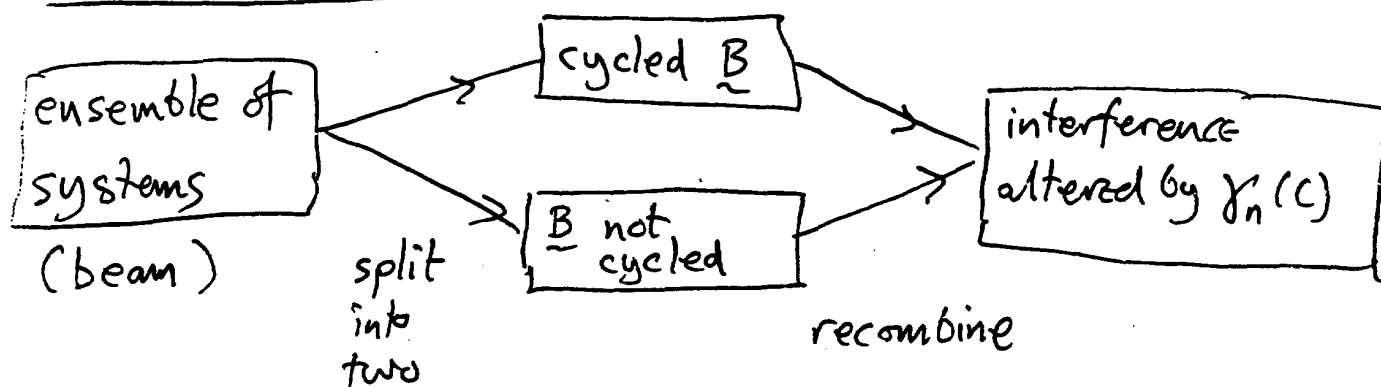
( $n$  = integer for bosons ( $\alpha$  particles,  
photons)  
= half-integer for Fermions  
(electrons, neutrons))

Geometric phase is

$$\gamma_n(C) = -n \Omega(C)$$

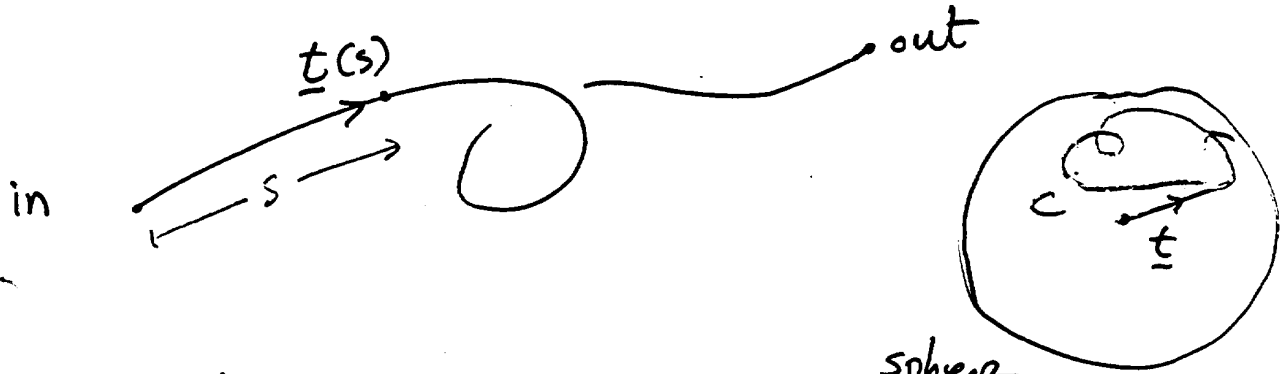
solid angle subtended by  $C$  at  $\underline{B} = 0$

Schematic experiment



# PHOTONS spin $n = \pm 1$ (Chiao-Wu-Tomita) 1.10

Photons don't interact with electric or magnetic fields, but their spins are parallel to their direction of propagation  $\underline{t}$ , and so can be turned by changing  $\underline{t}$ , e.g. by coiling an optical fibre along which the light is travelling.



phase shifts

sphere  
of unit tangents  $\underline{t}$

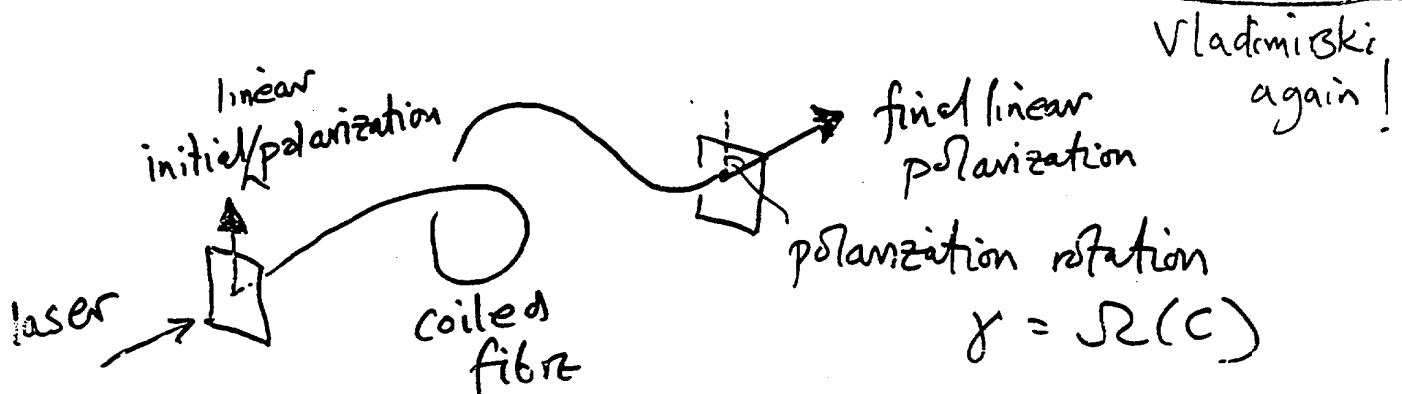
$$\begin{aligned} \gamma &= -\Omega(C) \quad (\text{spin } +1 : \text{right circular polarization}) \\ &+ \Omega(C) \quad (\text{spin } -1 : \text{left circular polarization}) \end{aligned}$$

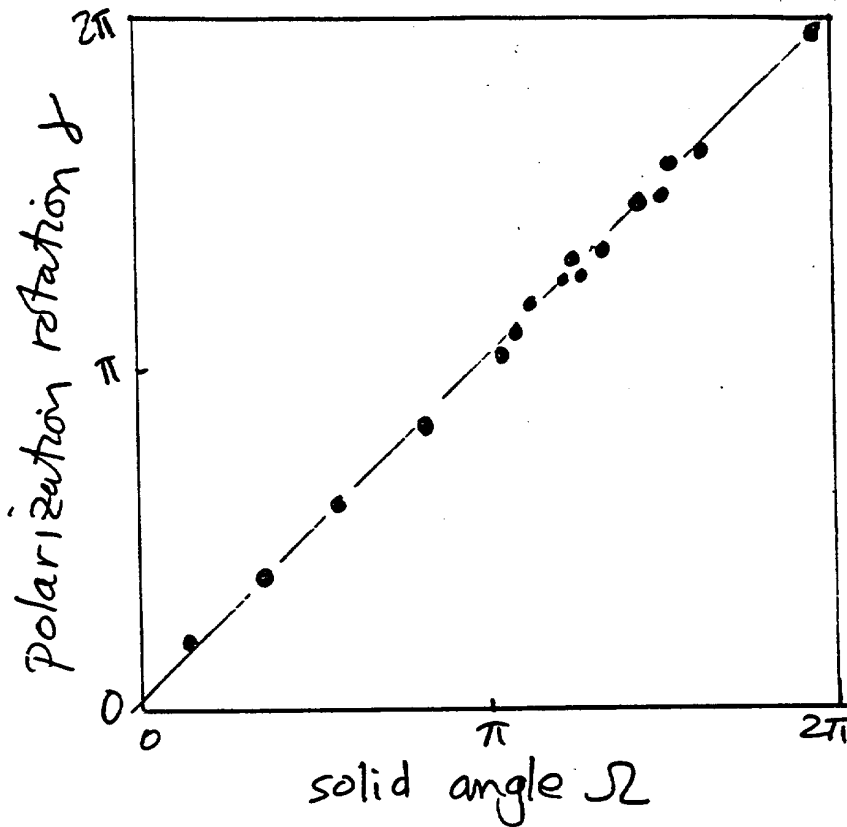
Instead of an interference experiment, it is easier to send in linearly polarized light:

linear polarization = superposition of R circular + L circular

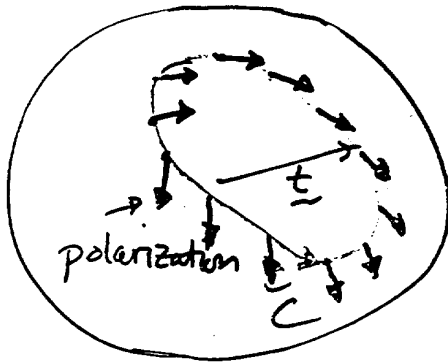
afterwards:

$$e^{-i\Omega} \times R + e^{+i\Omega} \times L = \text{linear polarization rotated by } \Omega$$





This is  
'geometric optical  
activity'  
the solid-angle  
dependence means  
that the polarization  
direction is  
parallel-transported  
over the  $\underline{t}$  sphere



(Ross 1984)

A natural question Is the effect quantum or classical?  
Chiao and Wu:

"... would rather think of these effects as topological features of classical Maxwell theory which originate at the quantum level, but survive the correspondence-principle limit ( $\hbar \rightarrow 0$ ) into the classical level"

(They also survive the short-wave limit ( $\lambda \rightarrow 0$ ) into geometrical optics - Rytov 1938, Vladimirovsky 1941)

Feynman | "The photon equation... is just the same as Maxwell's eqn"

ray arguments strictly not applicable in monomode fibres; need full Maxwell equations.

1.12

Quantum or classical?

Pseudoproblem because (Feynman)

The photon equation is just the same as Maxwell's equations ...

Maxwell

$$\partial_t (\text{Fields}) = (\text{matrix linear in } \nabla) (\text{fields})$$

$\times i\hbar$   $\times i\hbar$

$$i\hbar \partial_t (\text{Fields}) = (\text{matrix linear in } \underline{p} = -i\hbar \nabla) (\text{fields})$$

→ Schrödinger equation if matrix is Hermitian

Connection with spin:

$$-i\hbar \nabla \wedge \underline{E} = \underline{p} \wedge \underline{E} = -i(\underline{p} \cdot \underline{\epsilon}) \underline{E}$$

$$-i(\underline{p} \cdot \underline{\epsilon}) \underline{E} = \begin{pmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$\underline{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Spin 1 matrices

Explicitly:

1.13

$$\partial_t \underline{D} = \nabla \wedge \underline{H} ; \partial_t \underline{B} = -\nabla \wedge \underline{E} ; \underline{B} = \mu(r) \underline{H} ; \underline{D} = \epsilon(r) \underline{E}$$

fibre refractive index

$$i\hbar \partial_t |\underline{\psi}\rangle = \hat{H} |\underline{\psi}\rangle$$

$$|\underline{\psi}\rangle = \begin{pmatrix} \epsilon^{1/2}(r) \underline{E}(r,t) + i\mu^{1/2}(r) \underline{H}(r,t) \\ \epsilon^{1/2}(r) \underline{E}(r,t) - i\mu^{1/2}(r) \underline{H}(r,t) \end{pmatrix}$$

$$\hat{H} = c \begin{pmatrix} \underline{\Pi} \cdot \underline{\epsilon} & i\hbar \underline{\xi} \cdot \underline{\epsilon} \\ -i\hbar \underline{\xi} \cdot \underline{\epsilon} & -\underline{\Pi} \cdot \underline{\epsilon} \end{pmatrix}$$

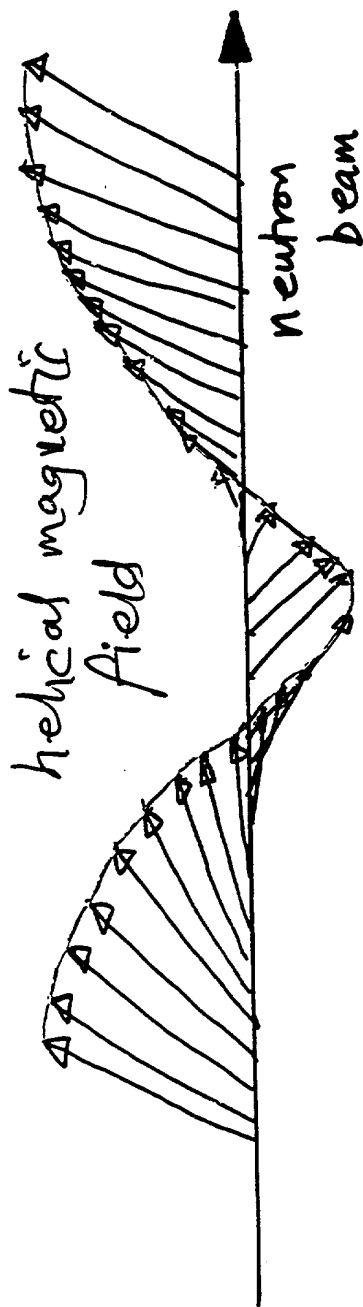
$$\underline{\Pi} = n^{+1/2}(r) \underline{p} n^{1/2}(r)$$

$$n(r) = \sqrt{\mu(r)\epsilon(r)}$$

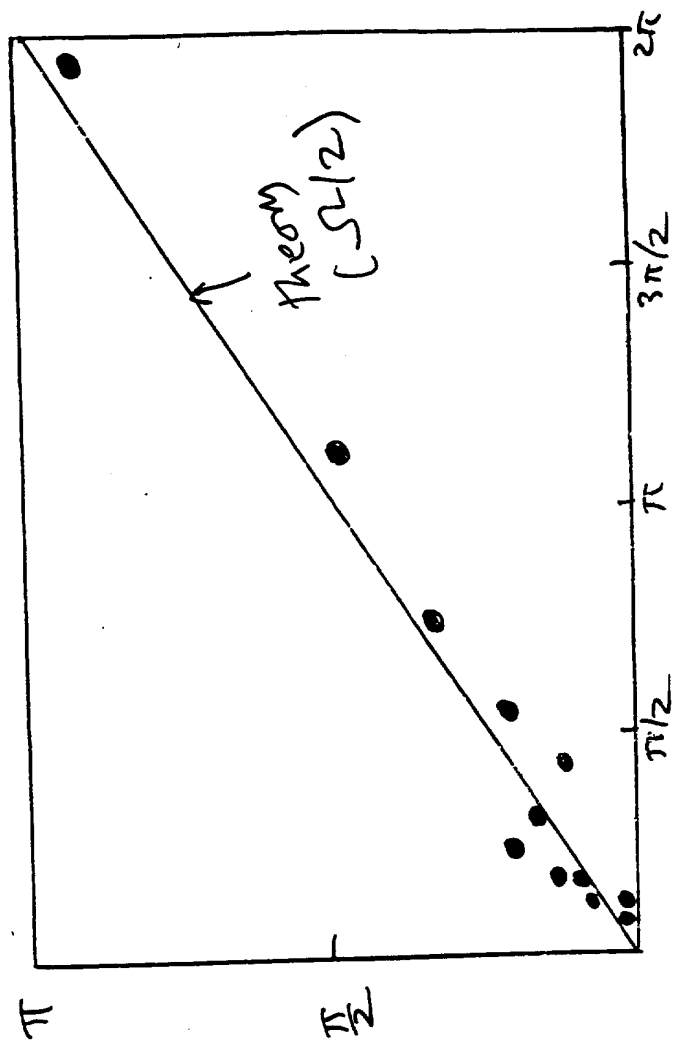
$$\underline{\xi}(r) = \frac{1}{4n(r)} \nabla \log \frac{\epsilon(r)}{\mu(r)}$$

So, Maxwell's equations (1865?), not only relativistic but also quantum mechanical!

# NEUTRONS, $n = 1/2$



T. Bitter and  
D. Dubbers

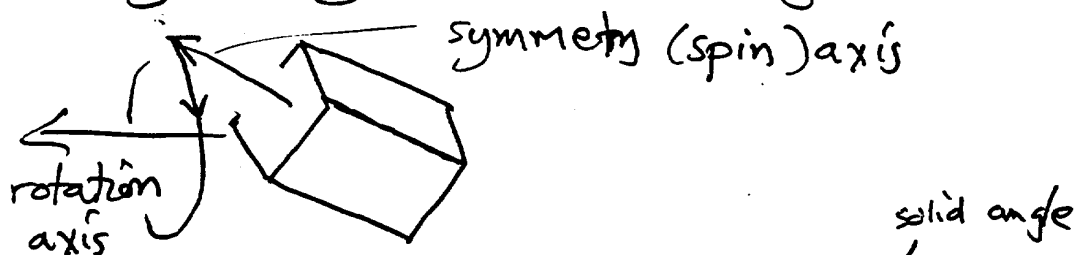




NUCLEI spin  $n = \pm 3/2$

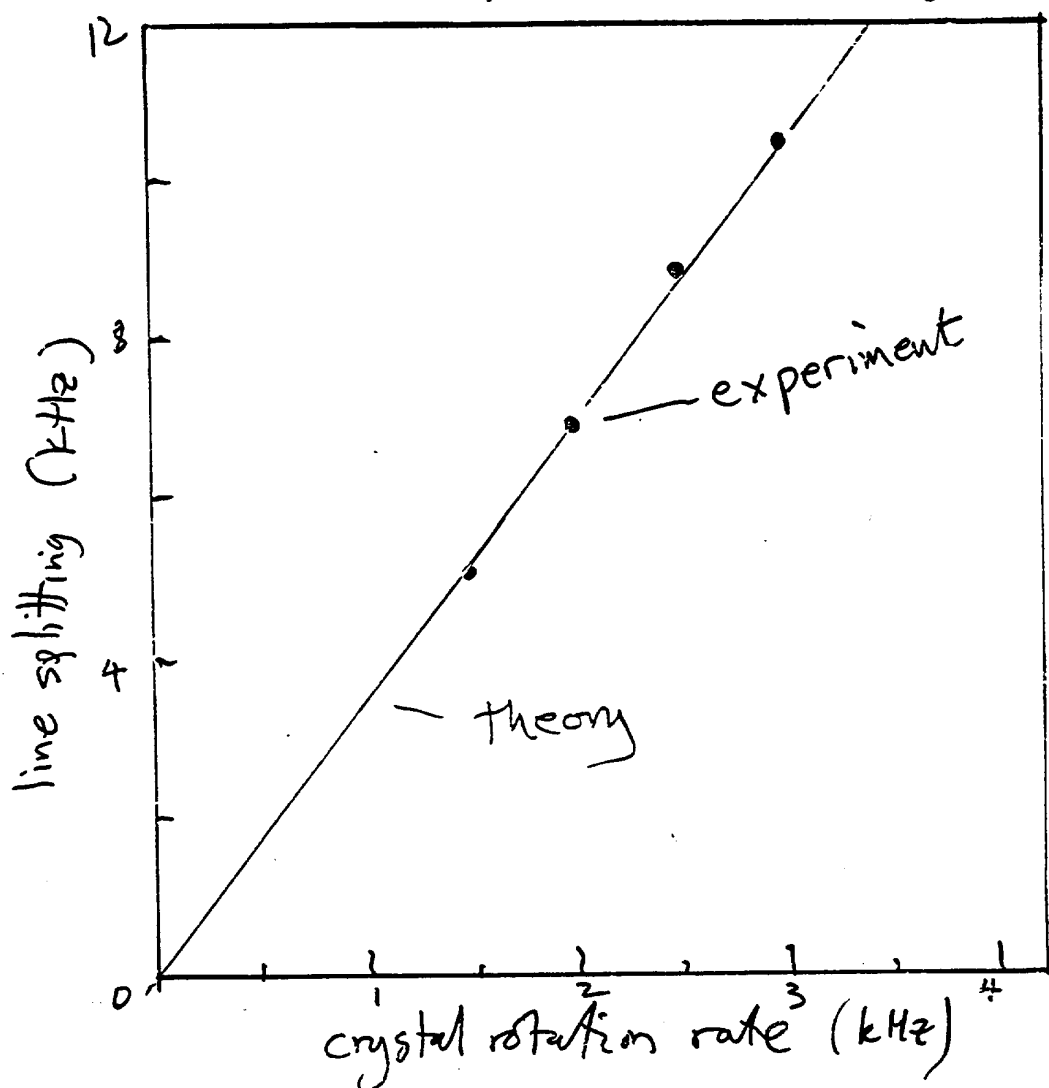
5.15

$^{35}\text{Cl}$  nuclei in crystalline  $\text{NaClO}_3$   
Spin has electric (quadrupole) interaction with  
crystal electrons, and so can be turned by rotating  
a symmetry axis of the crystal



Phase of state increases by  $\pm \frac{3}{2} \Omega$  per rotation.

Ever-changing phase  $\rightarrow$  frequency shift  $\left(\frac{d\text{phase}}{dt}\right)$   
measurable as line splitting in nuclear magnetic resonance



R. Tycko

# Geometric contribution to adiabatic transition probability

1.16

'Adiabatic transitions' are the exponentially weak transitions between states, governed by

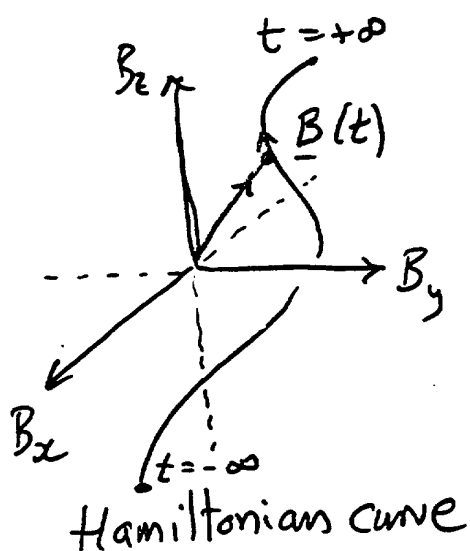
$$i\hbar \frac{\partial |\Psi\rangle}{\partial t} = \hat{H}(\delta t) |\Psi\rangle$$

when  $\hat{H}(\delta t)$  is non-degenerate and  $\delta$  is small

Example: Spin- $1/2$  in a  $\underline{B}$  field

$$i\hbar \frac{\partial |\underline{\Psi}\rangle}{\partial t} = \mu \underline{B}(\delta t) \cdot \underline{\sigma} |\underline{\Psi}\rangle$$

Pauli



Separation of adiabatic (instantaneous) eigenstates is  $2\mu |\underline{B}(\delta t)|$  - never zero for real  $t$ , so transitions arise from complex degeneracies

$$B_x^2(z_c) + B_y^2(z_c) + B_z^2(z_c) = 0$$

Old theory: Transition probability

$$P_{+ \rightarrow -} \sim e^{-\frac{4}{\hbar \delta} \text{Im} \int_{\gamma} dz B(z)} = e^{-\Gamma_d}$$

any real time

Old theory wrong!

1.17

$$P_{+ \rightarrow -}(\infty) = e^{-\Gamma_d} e^{\Gamma_g}$$

dynamical exponent  
(old)

$$\Gamma_d = \frac{4}{t\delta} \text{Im} \int_{\tau_0}^{\tau} dz B(z)$$

geometric exponent  
(new) - from // transport of adiabatic states

$$\Gamma_g = -2 \text{Im} \int_{\tau_0}^{\tau} dz \phi' \cos \theta$$

This is purely geometric  
- independent of  $\delta$  and  $t$ .  
 $\Gamma_g$  is a complex solid angle.

Example: Twisted  
Landau - Zener:

$$X_3 = A \tau$$

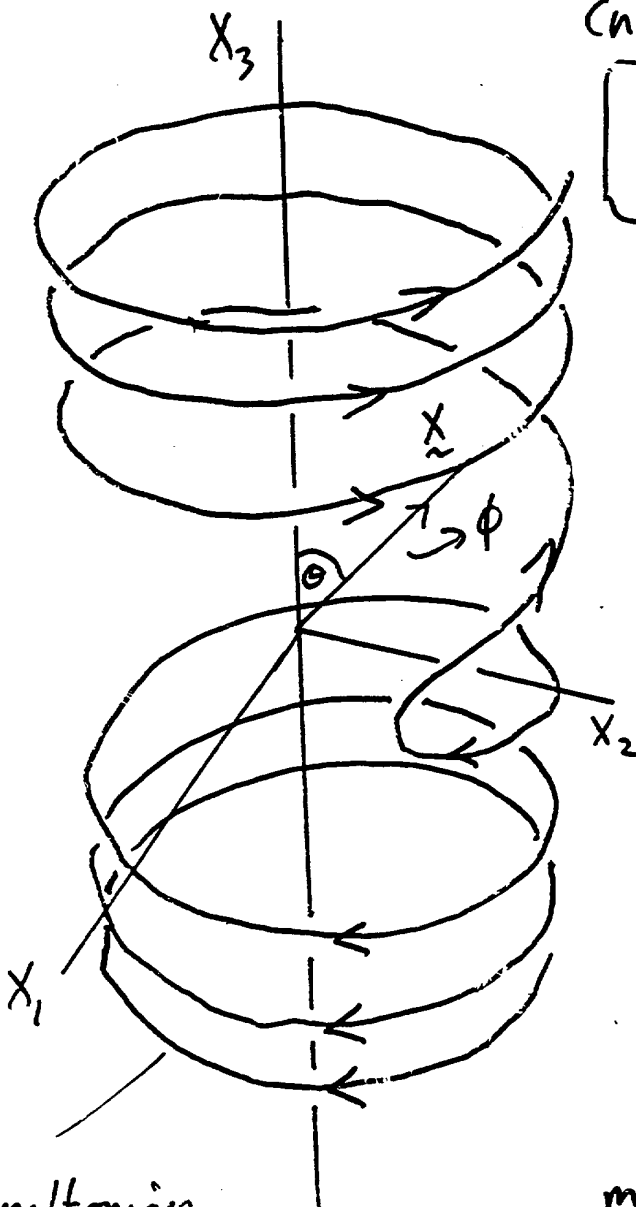
$$X_1 = \Delta \cos B \tau^2$$

$$X_2 = \Delta \sin B \tau^2$$

$$\Gamma_g = - \frac{\Delta^2}{A^2} B \text{sgn } A$$

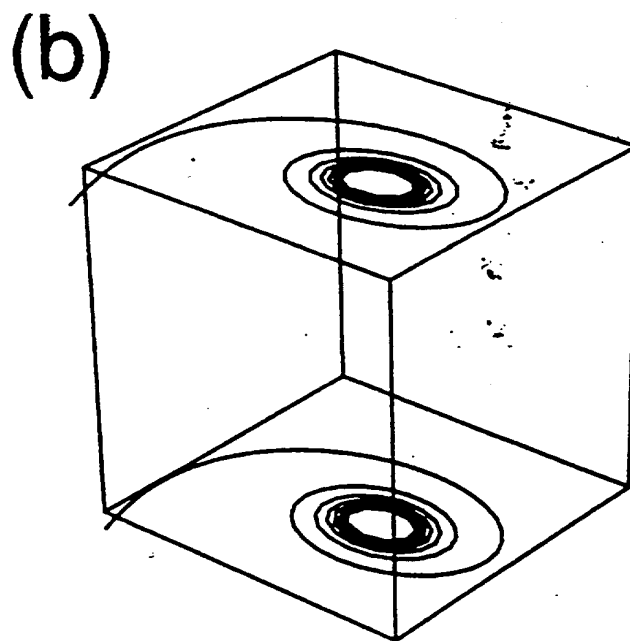
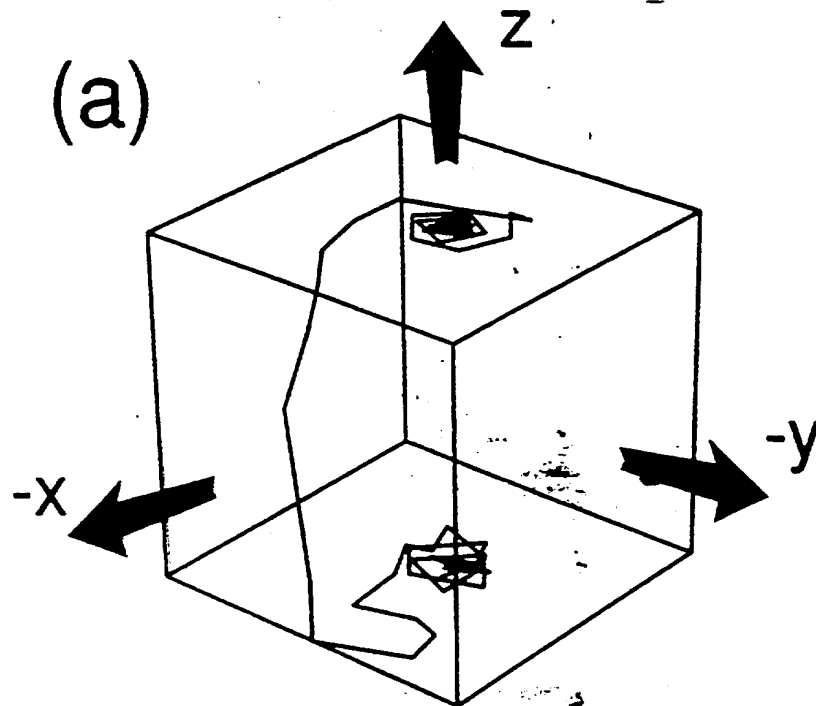
measurable in a spin experiment

MVB Proc Roy Soc 1990 A430 405-411



Hamiltonian  
curve

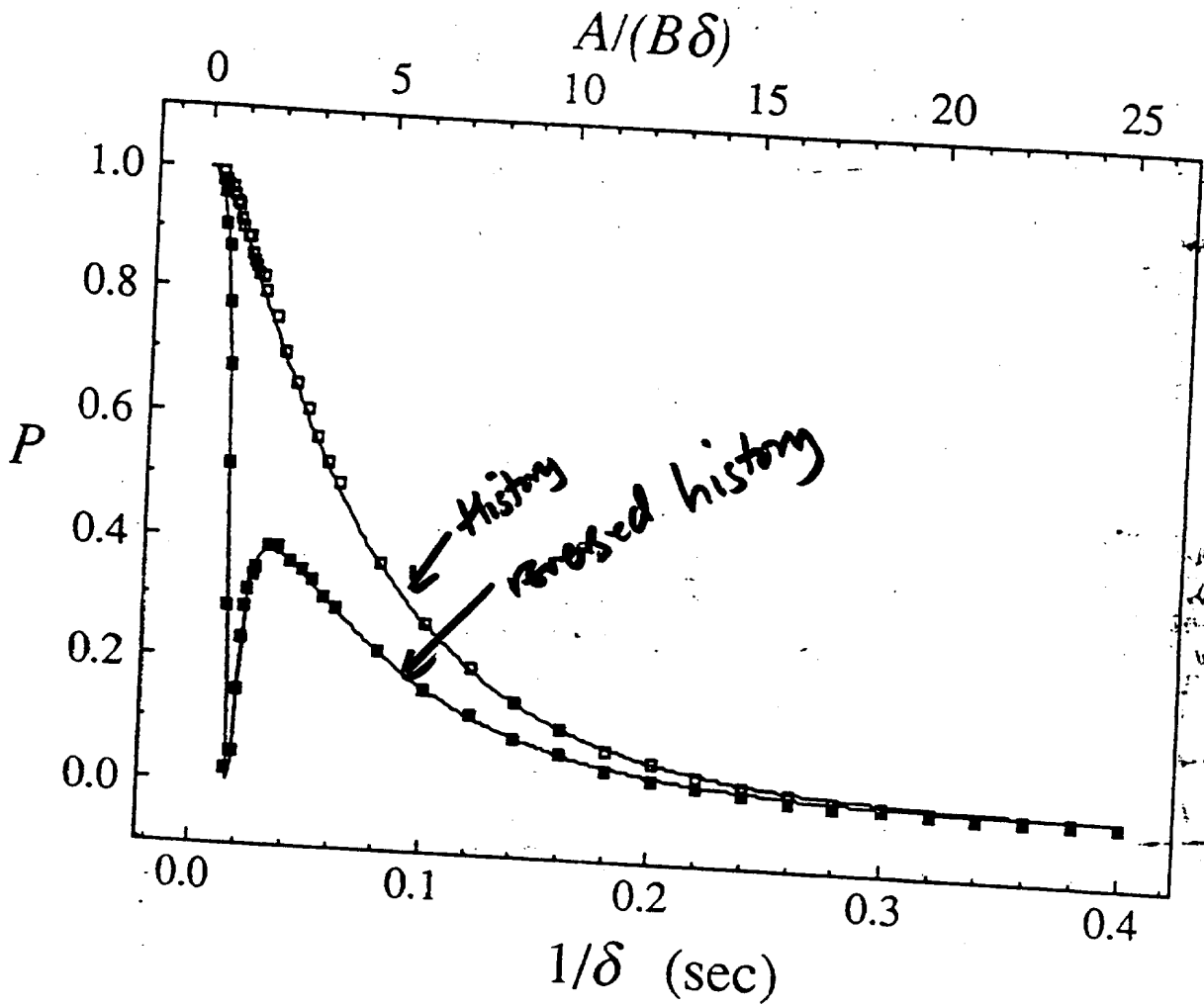
# Histories in Hamiltonian space



J.W. Zwanziger, S.P. Rucker and  
G.C. Chingas

Phys Rev A 43 (1991) 3232-9  
figure 3

NMR in  $^{13}\text{C}$



Predict

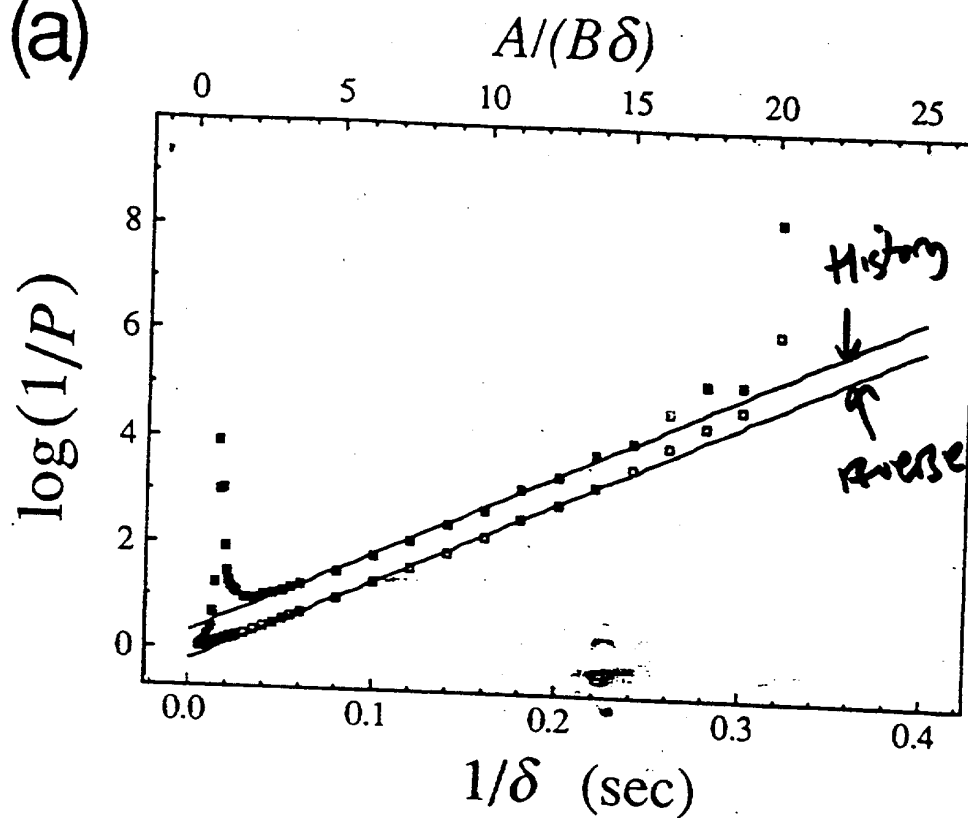
$$P_{+ \rightarrow -} = e^{-\frac{\Gamma_g - \Gamma_d}{\frac{1}{\delta \hbar} \int \ln(E_1 - E_2) dE}}$$

$O(\delta)$

under reversal,  $\Gamma_g$  reverses  
 $\Gamma_d$  does not

Figure 4

(a)



(b)

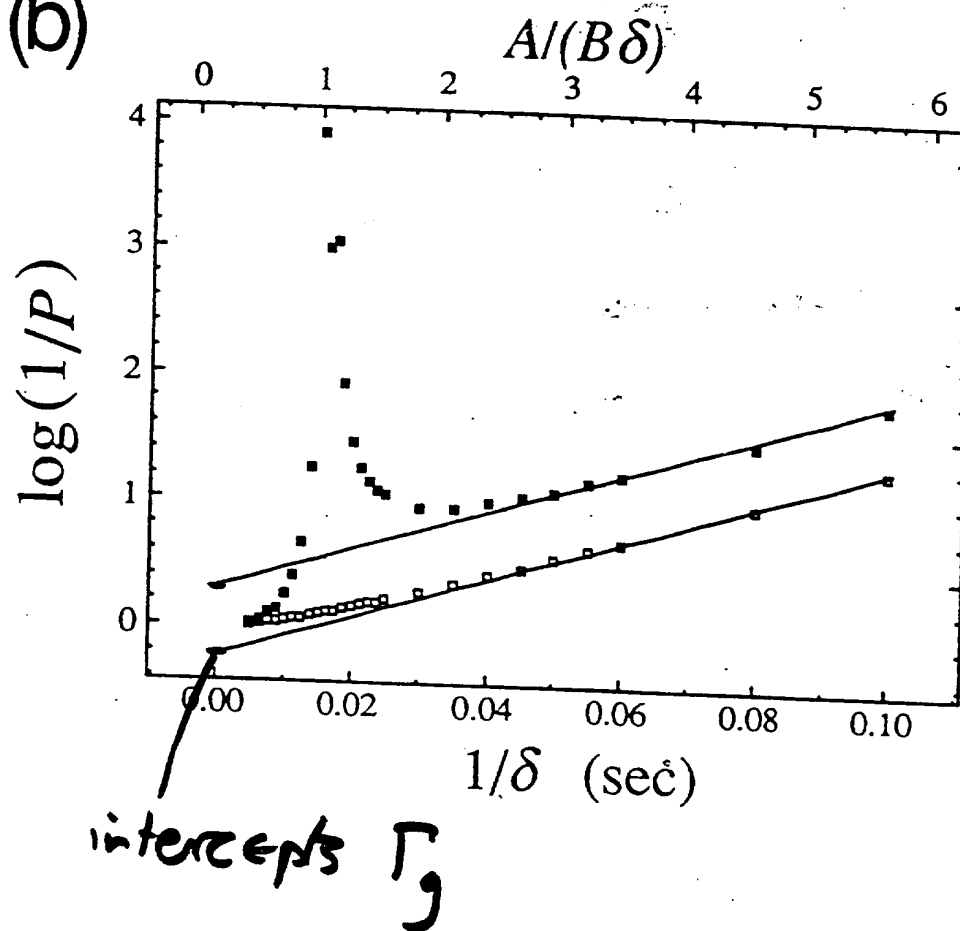
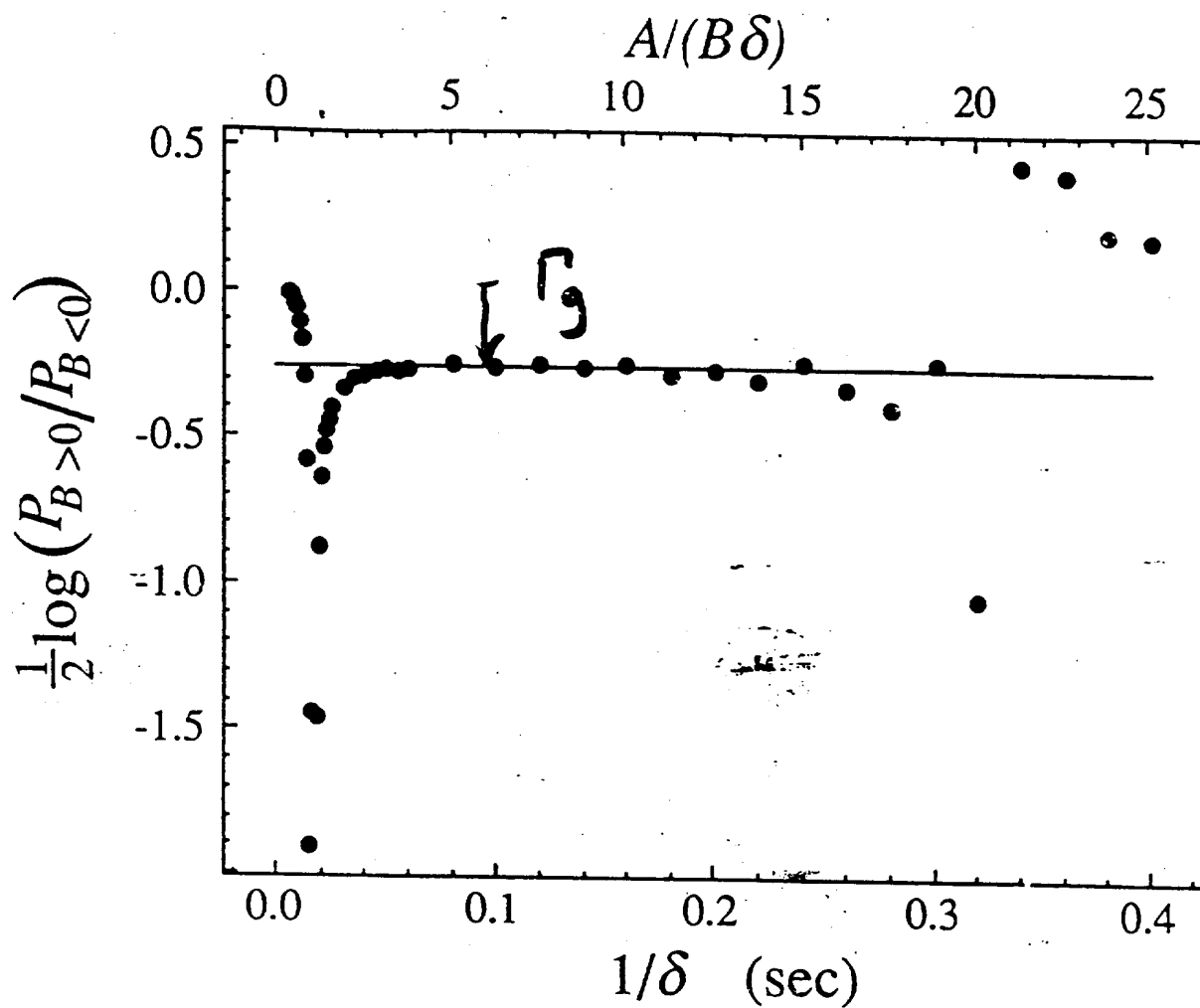


Figure 5



Theory  $\Gamma_g = -0.243$

Expt  $\Gamma_g = -0.26 \pm 0.01$

Figure 6

# Lecture 2

(Some geometric phases, Michael Berry Geneva 1993)



# BRISTOL ANHOLONOMY CALENDAR

## 1. Crystal dislocations as anholonomy (Frank 1951)

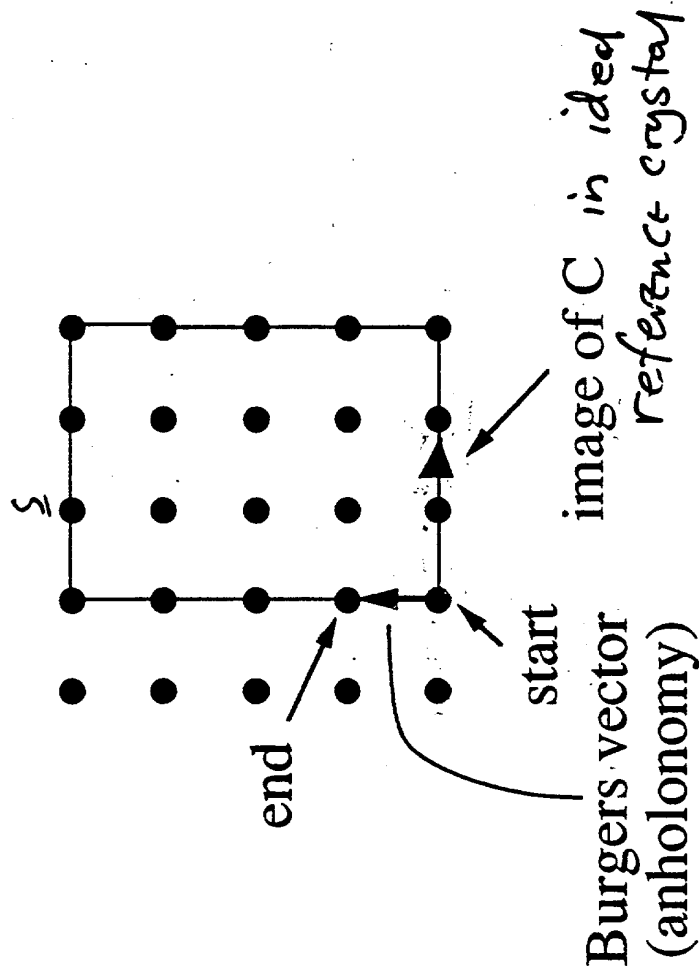
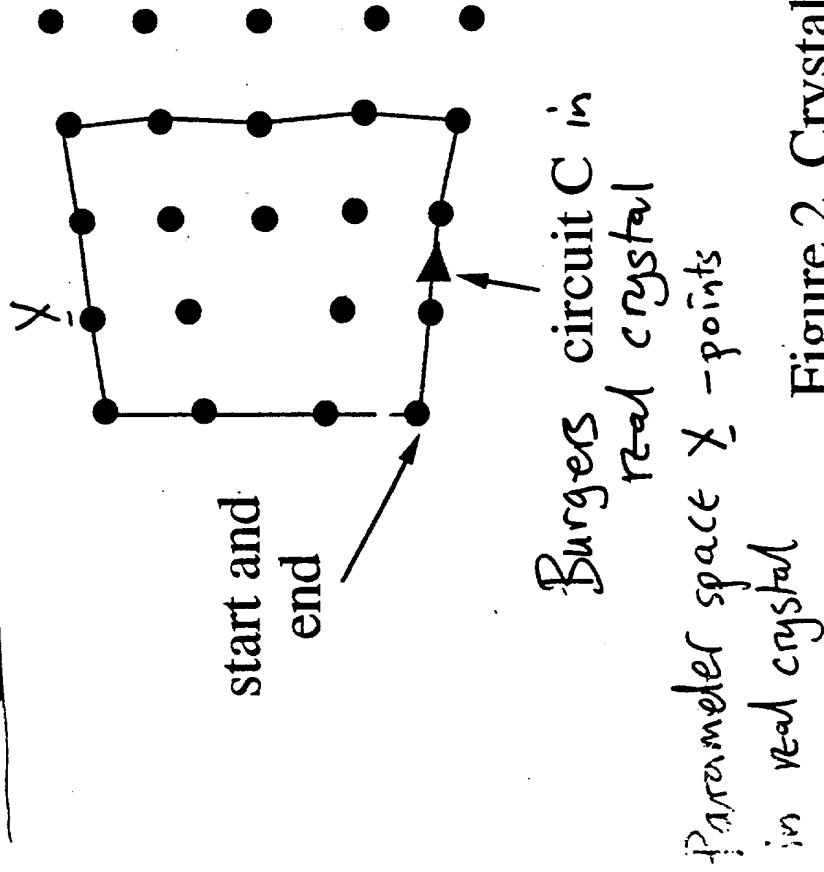
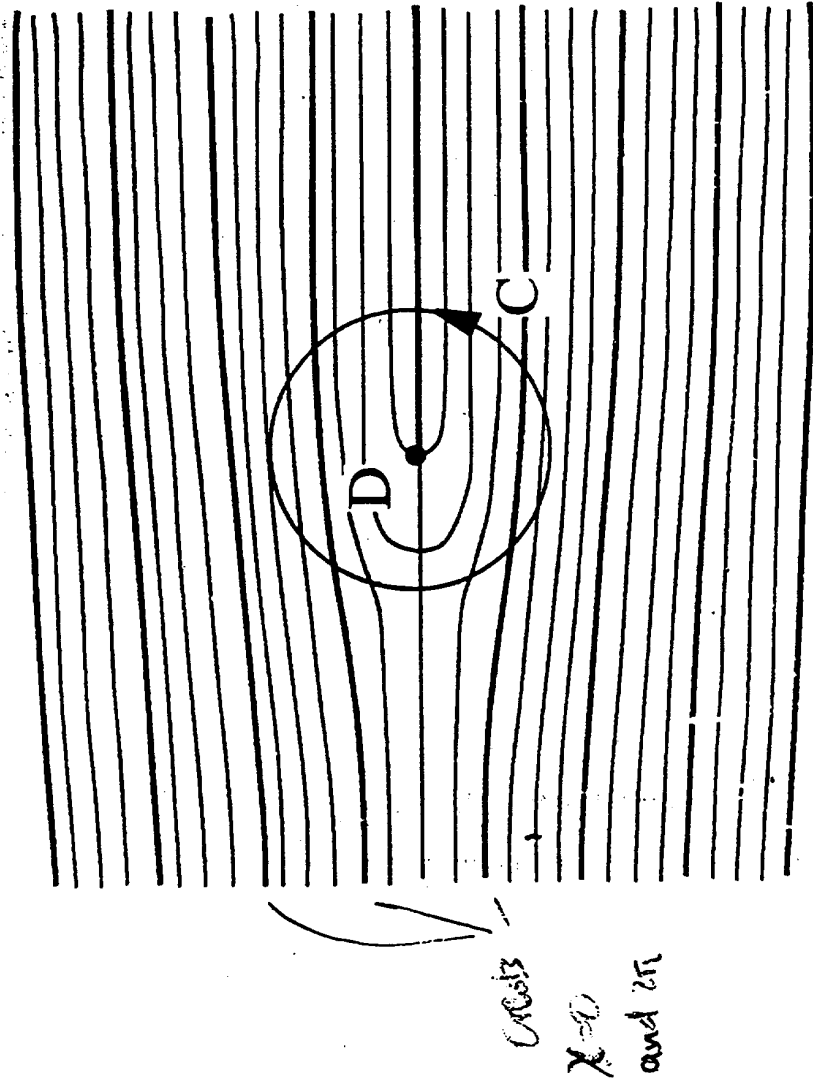


Figure 2. Crystal dislocation as anholonomy

Slaved variables  
 $\underline{S}$  - points in  
 ideal crystal

## 2 Dislocations in waves (Nye and Berry, 1974)



Wave  $\psi(x) = p(x) e^{i\chi(x)}$   
 wavefronts  
 $\chi(x) = \text{constant mod } 2\pi$

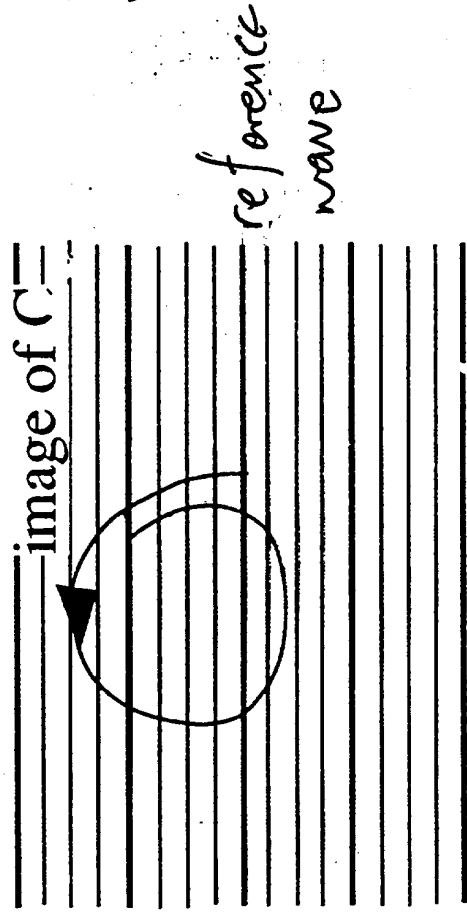


Figure 5. Wavefronts (at intervals of  $\pi/4$  with crests bold) of a dislocated wave with a Burgers circuit surrounding the dislocation D, and its image in a plane wave

At dislocations,  $\chi$  has a  $2\pi$  anholonomy, and  $\rho$  is zero  
 threads of silence  
 sound, microwaves, light, tides,  $N$

$\frac{1}{2\pi} \oint d\chi = \text{integer}$

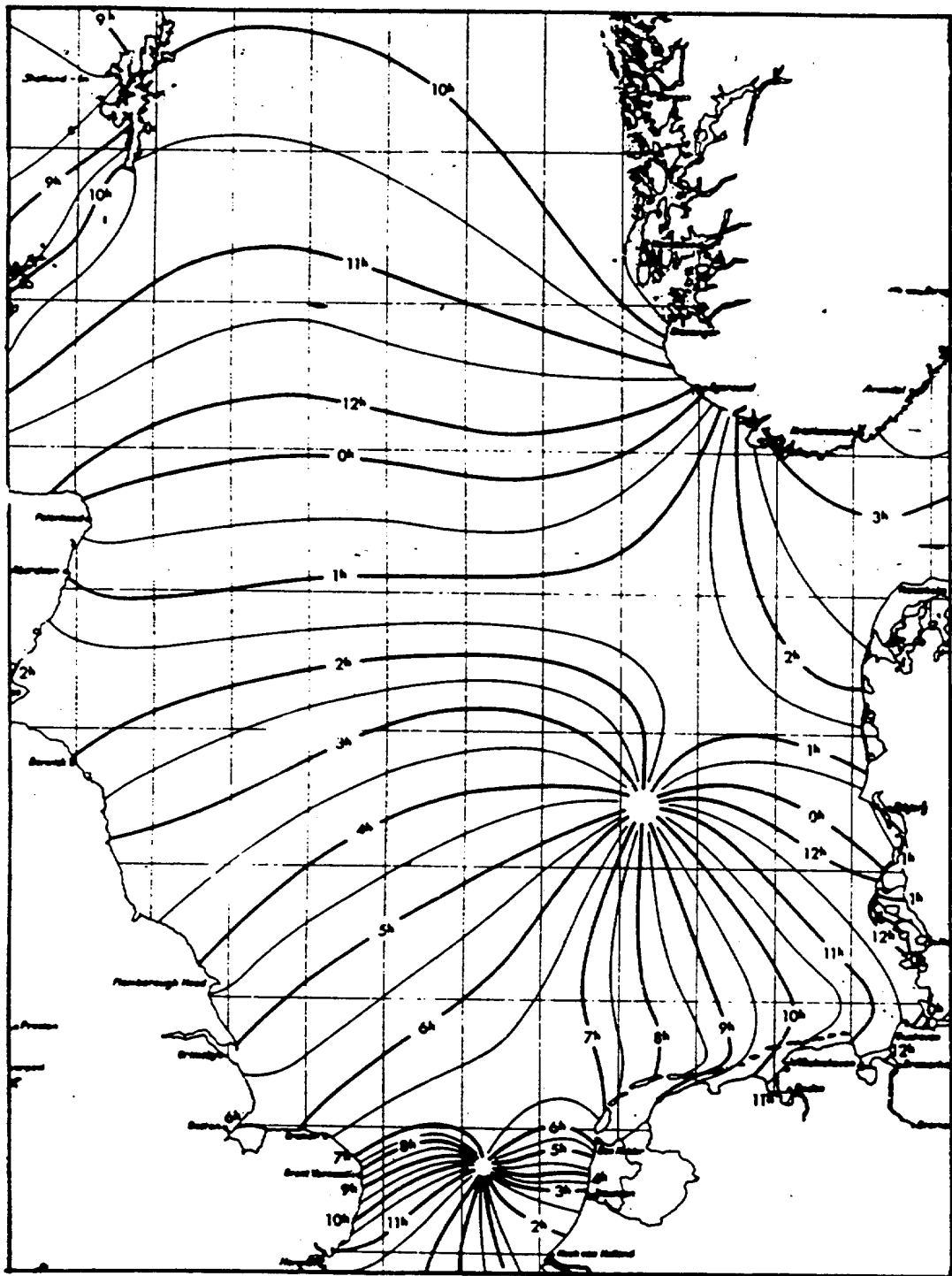
Dislocations are lines

## Dislocations in the tide wave (Whewell (1833))

2.3

Wavefronts : loci of instantaneous high tides at given times

Dislocations : 'amphidromic points' of no tide



The tide wave (12h period) is a forced vibration of the whole earth. Rotating source  $\rightarrow$  no time-reversal symmetry  $\rightarrow$  complex waves (stationary but not standing)

$$h(\theta, \phi; t) = \text{Re} \left[ e^{-i\omega t} \psi(\theta, \phi) \right] \quad \text{complex}$$

3.

AHARONOV-BOHM EFFECT

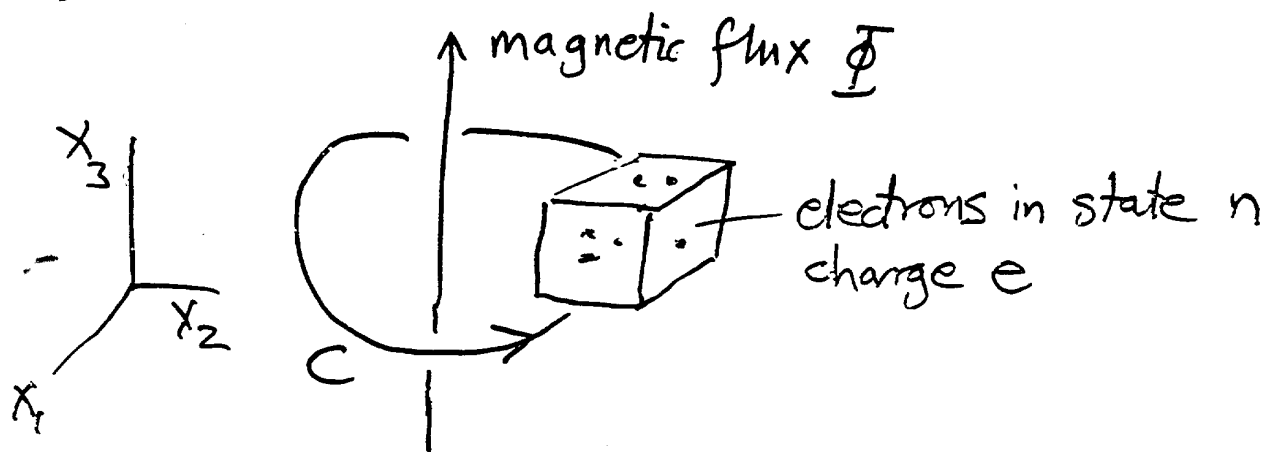
2.4

(1959)

System: electrons

environment: 3-D space of positions, containing a line of magnetic flux (e.g. long solenoid)

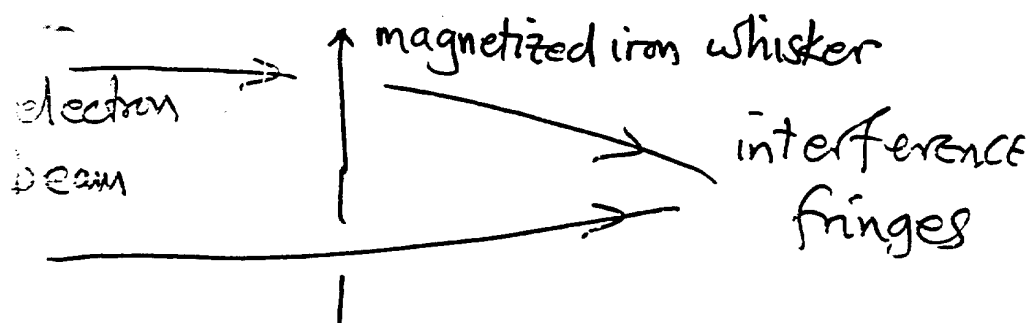
cycle: circuit around the flux line



After the circuit, geometric phase is

$$\gamma = e\Phi / \hbar$$

(electrons influenced quantumly by field elsewhere)

Chambers' experiment 1959The flux  $\Phi$  shifts the fringes, via an extra phase:

$$\begin{aligned} & \text{Phase}(\rightarrow) - \text{phase}(\searrow) \\ &= \text{phase}(\rightarrow) + \text{phase}(\swarrow) \\ &= \text{phase}(\bigcirc) = \text{geometric phase} \end{aligned}$$

# FOUR CLASSES OF AB EFFECT

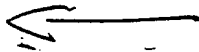
| Effect    | wave      | flux      |                                    |
|-----------|-----------|-----------|------------------------------------|
| AB        | quantum   | classical | electrons, mag flux                |
| Tomomura  | quantum   | quantum   | electrons on flux in superconduct  |
| Steinberg | classical | quantum   | sound waves on qubit vortex in the |
| Bathub    | classical | classical | ripples on bathub vortex           |

But, can observe water-wave analogue of AB effect <sup>2.6</sup>  
 For waves on a medium moving with speed  $\underline{v}(\underline{r})$  ( $\ll c$ ),  
 $\underline{v}(\underline{r})$  acts like a vector potential, so for AB need

$$\nabla_{\perp} \underline{v} = A \delta(\underline{r}) \text{, i.e. a vortex line in irrotational flow.}$$

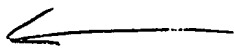
This can be realized with surface ripples scattered from a bathtub vortex

wave direction



strength

$$S=2$$



dimensionless flux

$$\alpha = \frac{\oint \underline{v} \cdot d\underline{\ell}}{\lambda c_{\text{group}}}$$

wavelength  
group vel.  
circulation

M.V.B., R.G. Chambers et. al, Eur. J. Phys 1 (1980) 154-162

Dislocations can be

- 1) unattached to any singularity of the medium (generic case)
- 2) coincident with line singularities (AB)
- 3) attached to a point singularity and otherwise movable (but not removable) by gauge transformation (Dirac string)

#### 4. Liquid crystal disclinations as anholonomy (Frank 1959)

2.7

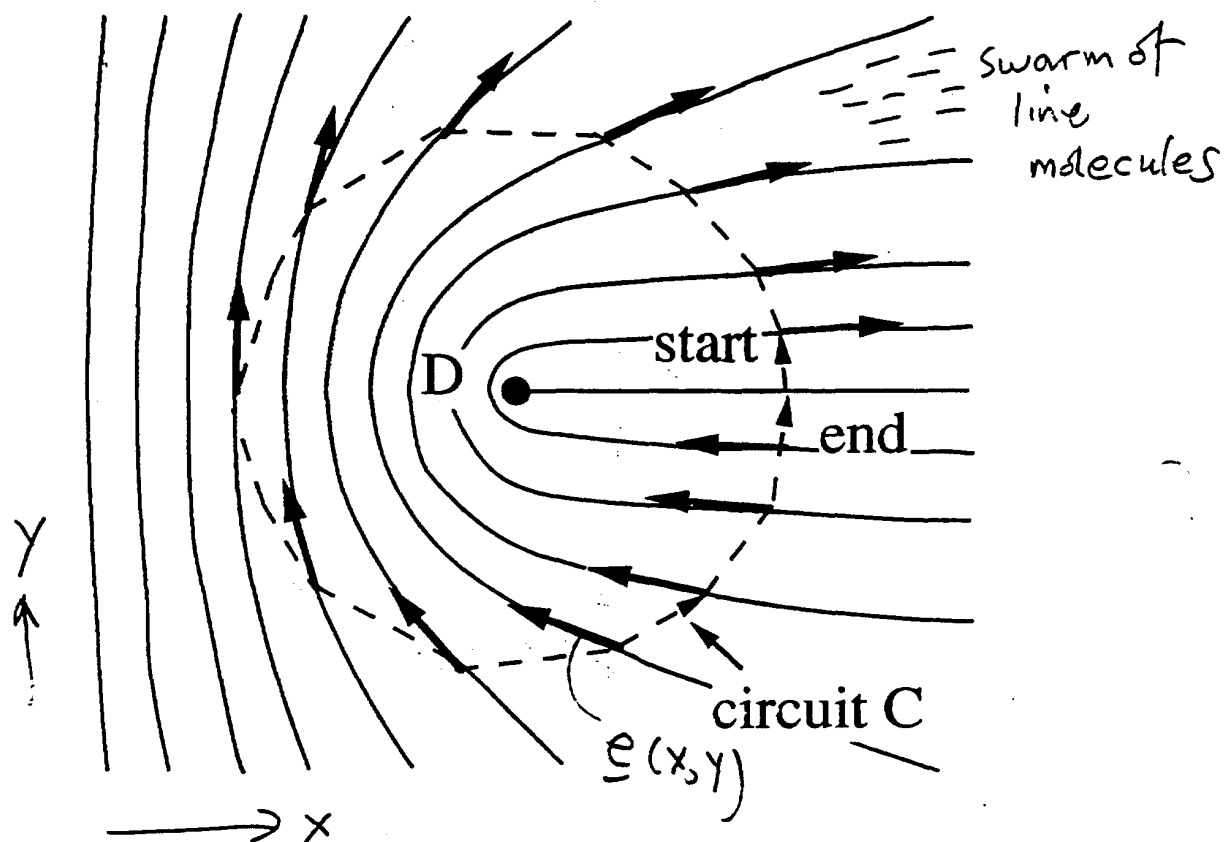


Figure 3. Reversal ( $\pi$  anholonomy) of liquid crystal direction around a disclination D

Can regard line field as field of eigendirections of a  $2 \times 2$  real symmetric matrix

$$M(x,y) = \begin{pmatrix} M_{11}(x,y) & M_{12}(x,y) \\ M_{12}(x,y) & M_{22}(x,y) \end{pmatrix}$$

$$\underline{M} \underline{e} = \lambda \underline{e}$$

Disclination D is a degeneracy  $M_{11} = M_{22}$ ,  $M_{12} = 0$   
 If C encloses D,  $\underline{e}$ 's change sign -  
 [Geometric phase of  $\pi$ ]

## Differential geometry of curved surfaces.



Curvature matrix

$$H(x, y) = \begin{pmatrix} \partial^2 f / \partial x^2 & \partial^2 f / \partial x \partial y \\ \partial^2 f / \partial y \partial x & \partial^2 f / \partial y^2 \end{pmatrix} \quad \begin{matrix} \text{real,} \\ \text{symmetric} \end{matrix}$$

parameters

Eigenvalues: principal curvatures at  $x, y$ Eigenvectors: (orthogonal) directions of principal curvatures at  $x, y$ .

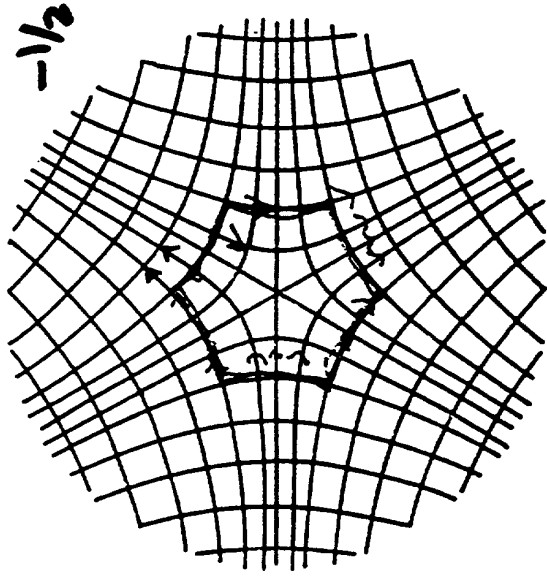
degeneracy: umbilic point on surface (locally spherical)

Sign change  $\rightarrow$  a line of curvature turns through  $\pi$  (reverses) in a circuit of an umbilic point

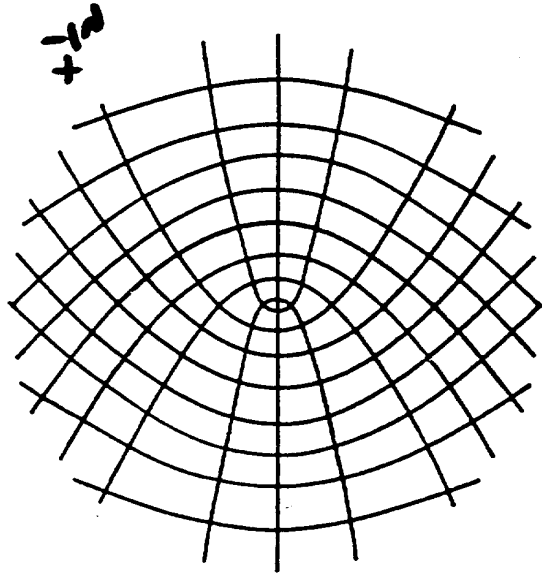


Curvature near an umbilic point of a surface

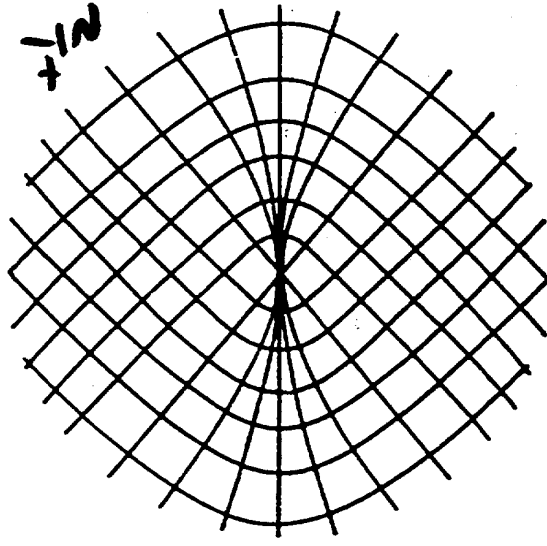
(surface:  $z(x, y)$ ; curvature matrix:  $\begin{pmatrix} \partial^2 z / \partial x^2 & \partial^2 z / \partial x \partial y \\ \partial^2 z / \partial y \partial x & \partial^2 z / \partial y^2 \end{pmatrix}$ ; eigenvalues: principal curvatures; eigenvectors: directions of principal curvature; degeneracy = umbilic point (locally spherical))



Star



Lemon



Monster

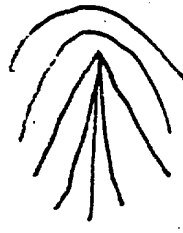
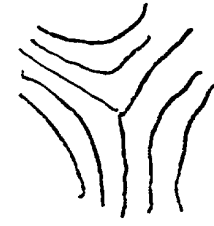


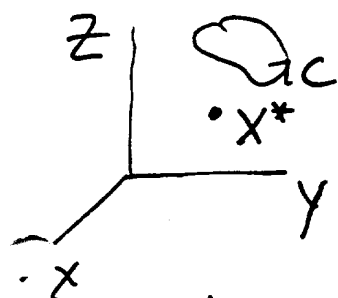
fig. 44

(disclinations of directors of liquid crystals)

(Berry and Hannay 1977)

Degeneracies These are singularities of the 2-form generating the phase

$$\gamma_n(C) = \iint_{\partial S=C} \text{Im} \langle dn | n | dn \rangle \equiv \iint_{\partial S=C} V_n(X)$$



Singularities of  $V_n$  at  $x^*$ , where  $|n\rangle$  (transported state) degenerates with  $|n+1\rangle$  or  $|n-1\rangle$

locally a 2-state system (like  $\text{spin} - \frac{1}{2}$ ), so can choose parameters in which Hamiltonian is, locally

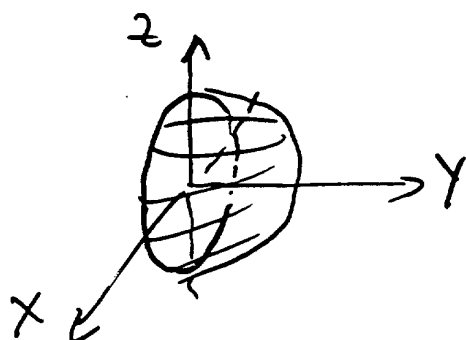
$$H = \frac{1}{2} \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \rightarrow V_n \approx \frac{R \cdot dS}{R^3}$$

monopole

Then  $\gamma_n(C) \approx \pm \frac{1}{2} \Omega(C)$  for  $C$  near  $x^*$

Important special case:  $H$  real (e.g. time-reversal symmetry)

i.e.  $y=0$ . Circuits in  $z$ - $x$  plane



$$\Omega = 2\pi \quad (\text{hemisphere})$$

$$\therefore \gamma = \pm \pi$$

$$e^{i\gamma} = -1$$

Sign change of real symmetric matrix round degeneracy

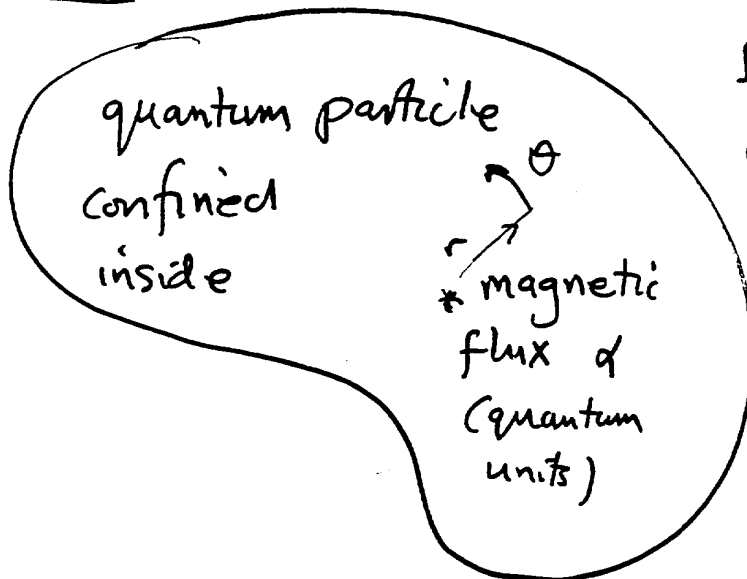
# Exploring monopoles in parameter space

2.11

with Raul Mondragon

Need a 3-parameter family of quantum systems, uncomplicated by any symmetries (e.g. time reversal)

Such a family is Aharonov-Bohm chaotic billiards



Boundary is a cubic conformal transformation of unit disc, i.e

$$W(z) = z + Bz^2 + Ce^{i\chi} z^3$$
$$|z| = 1$$

$$\text{Fix } \chi \equiv \pi/3$$

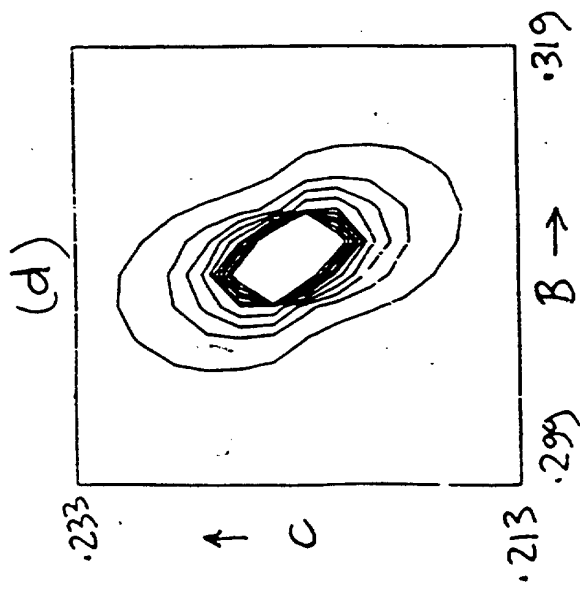
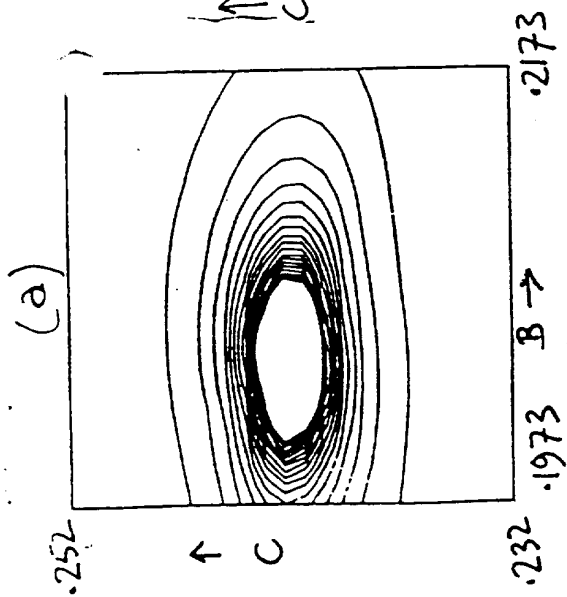
Hamiltonian

$$\hat{H} = -\left(\nabla - i\alpha \frac{\mathbf{e}_\theta}{r}\right)^2 ; \hat{H}|\psi\rangle = E|\psi\rangle$$

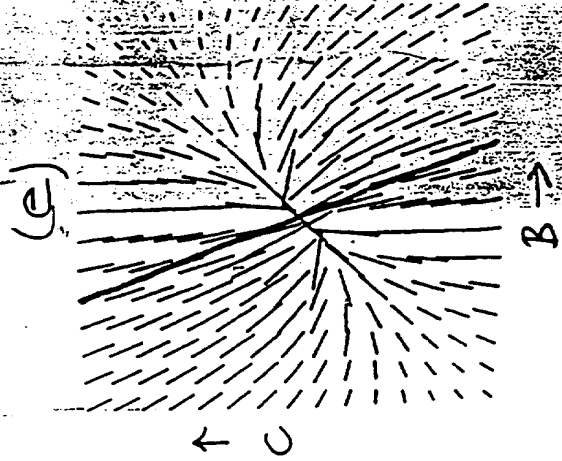
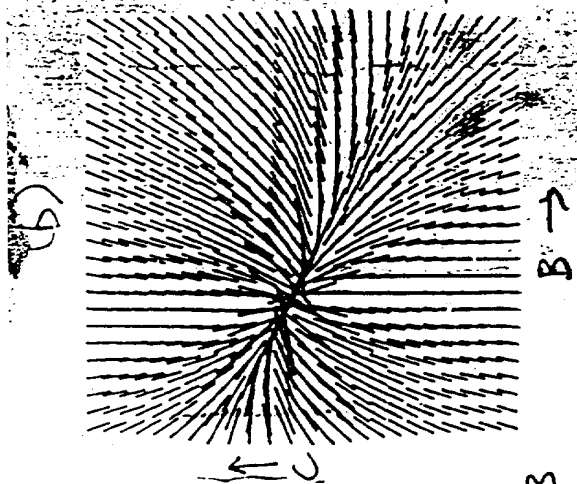
$$\psi(r) \equiv \langle r | \psi \rangle = 0 \text{ on Boundary}$$

$$\text{Parameters } (B, C, \alpha) \equiv \underline{\mathbb{R}}$$

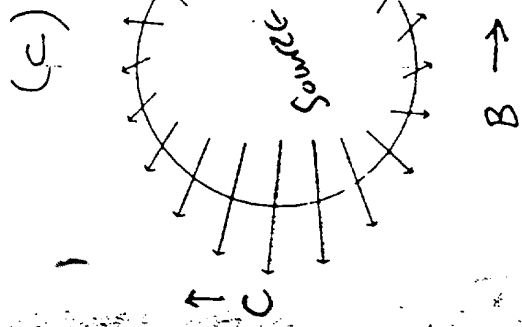
$$\underline{2\text{Form}} \quad \underline{V} = \text{Im} \langle \nabla_{\underline{R}} \psi_n | \wedge | \nabla_{\underline{B}} \psi_n \rangle$$



Contours of  $|V|$

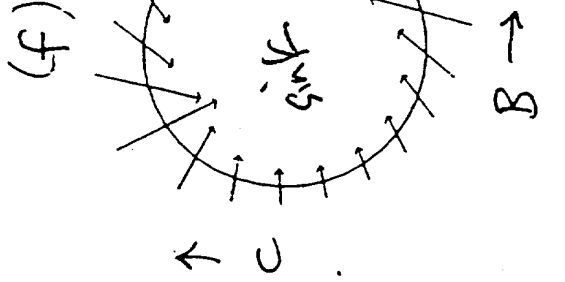


Lines of  $V$



$$R^* = \begin{pmatrix} .204 & .15 \\ .242 & .2 \\ 0 & .2 \end{pmatrix}$$

states  $|7\rangle$  and  $|8\rangle$



$$R^* = \begin{pmatrix} .309 & .2 \\ .223 & .2 \\ .382 & .2 \end{pmatrix}$$

states  $|14\rangle$  and  $|15\rangle$

Directions of  $V$  on a circuit of  $R^*$

Monopole behaviour of  $\underline{V}$  :

$$|\underline{V}| \propto \frac{1}{r^2}$$

where  $r = \sqrt{(B-B^*)^2 + (C-C^*)^2}$

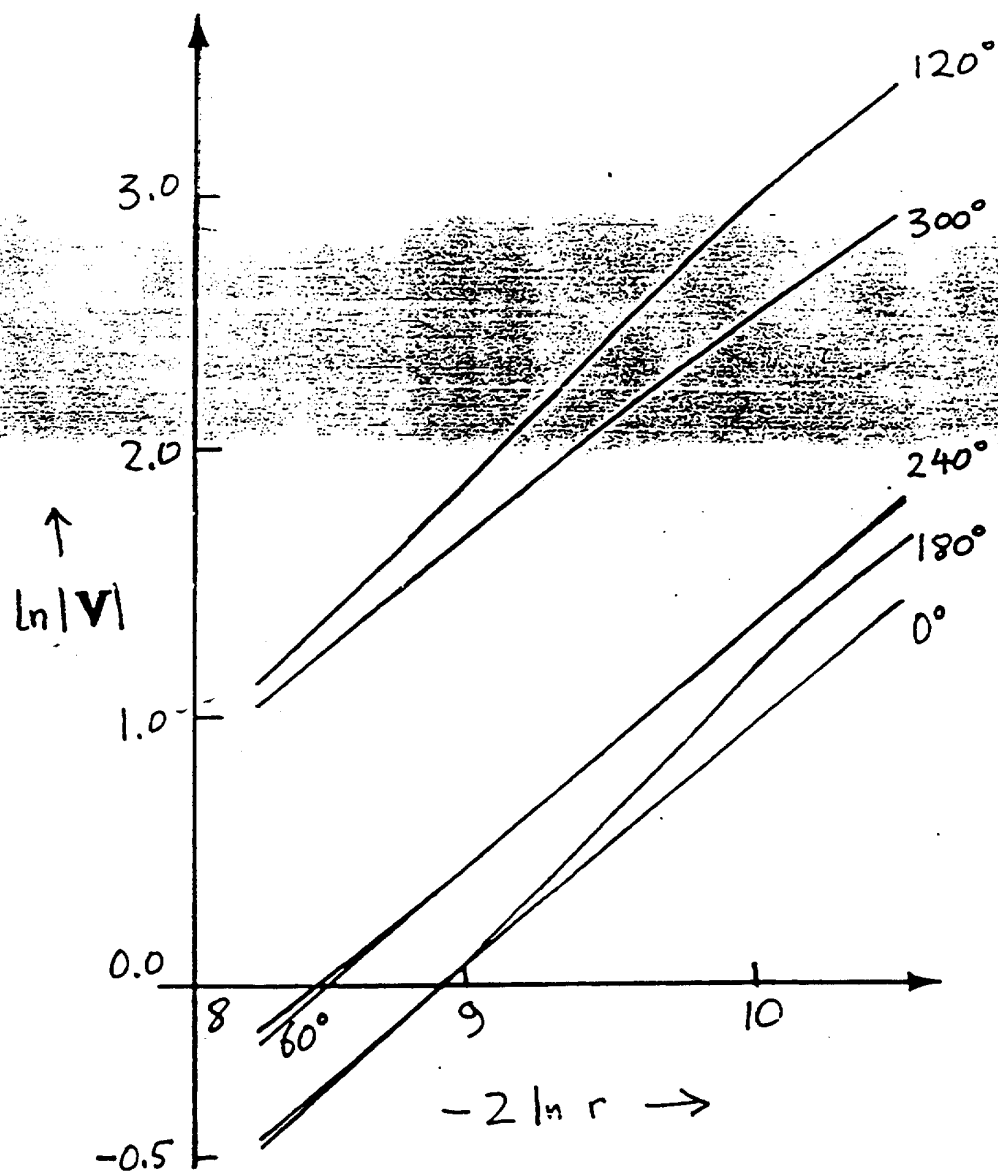
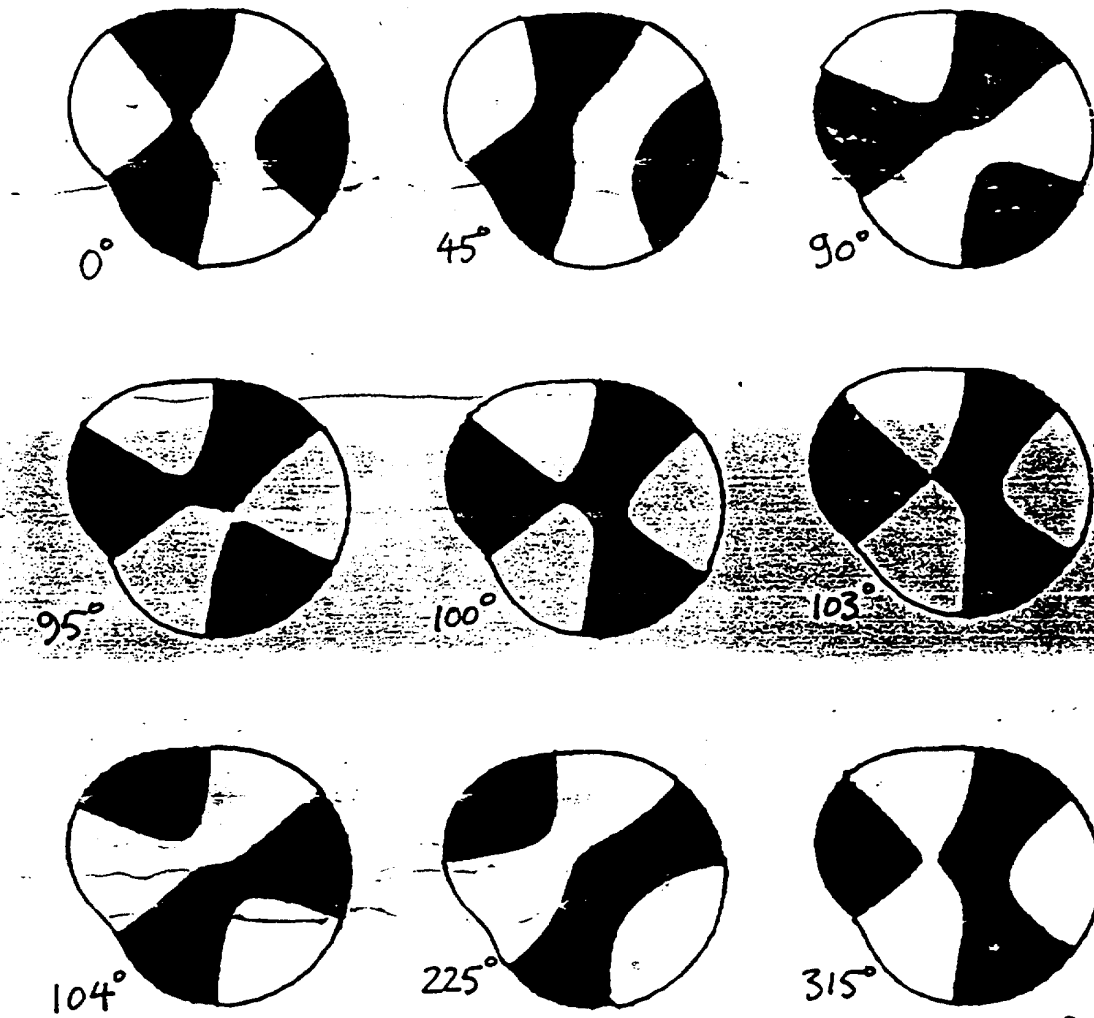


Figure 2

Rearrangement of nodal cells round degeneracy  
 at  $\underline{R}^* = (\underset{B}{.204} \quad \underset{C}{.242} \quad \underset{d}{0})$   $\therefore$  states real



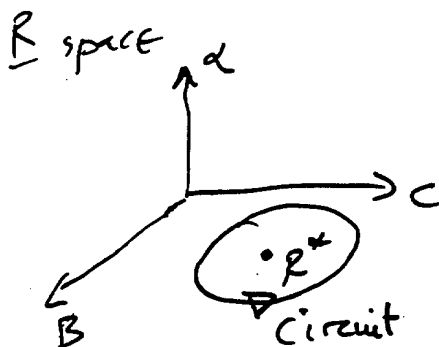
sign change

$$|\psi\rangle = \cos \frac{\theta}{2} |7\rangle + \sin \frac{\theta}{2} |8\rangle$$

$$0 \leq \theta \leq 2\pi$$

Figure 7

two states at  $\underline{R}^*$



# Singularities within singularities

Contours of phase  $\chi$  of complex wavefunction

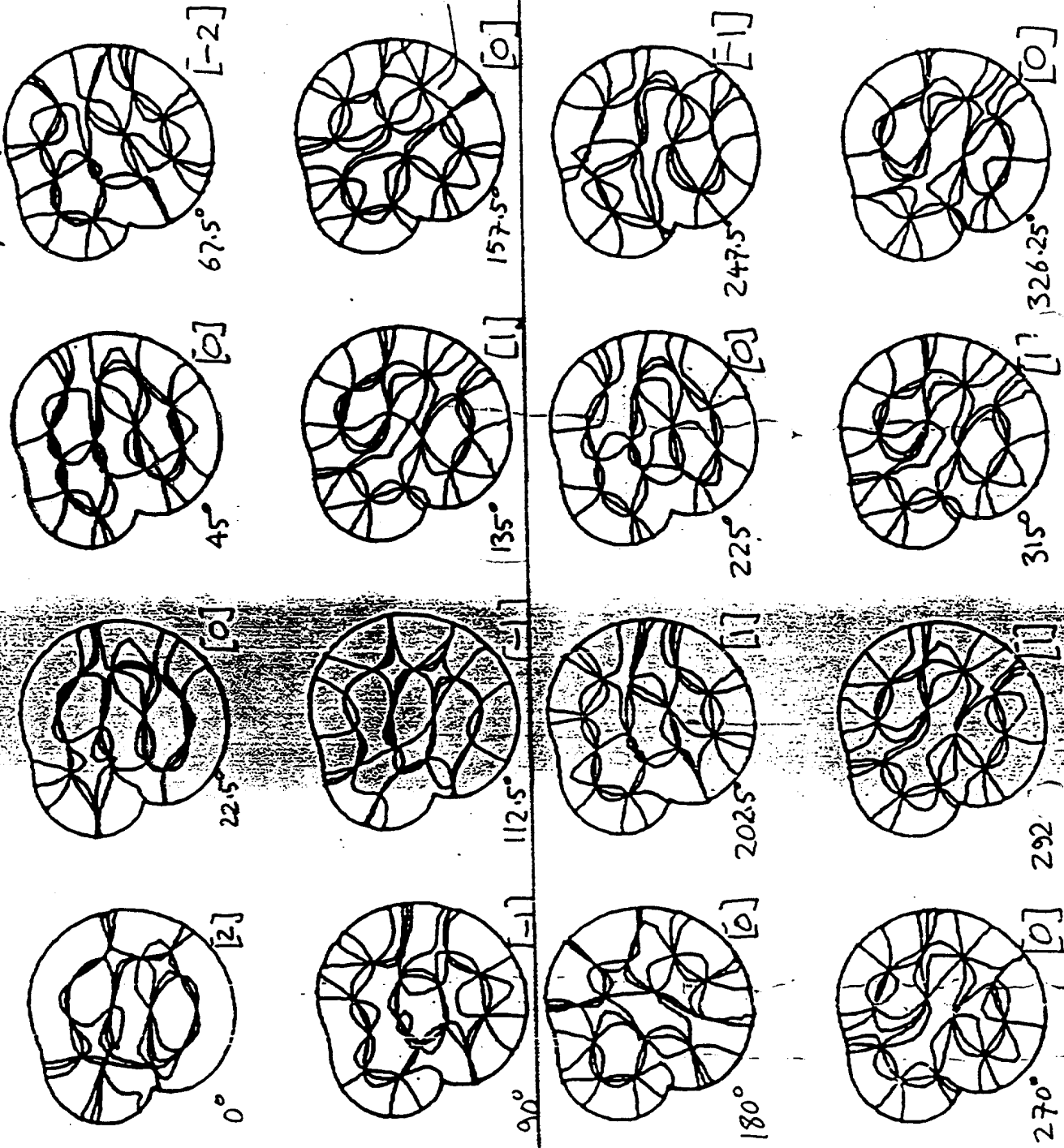
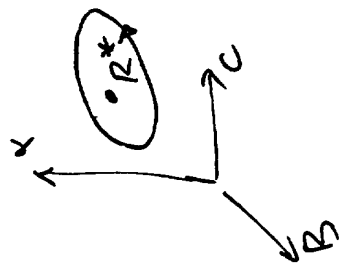
$$\psi(r) = \rho(r) \times e^{i\chi(r)}$$

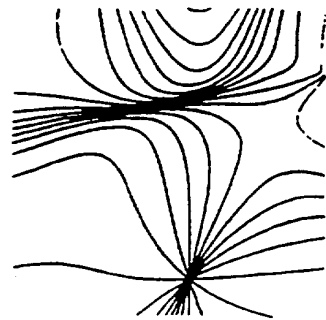
round a circuit of degeneracy at

$$R^* = \begin{pmatrix} .309 \\ .223 \\ .382 \end{pmatrix}$$

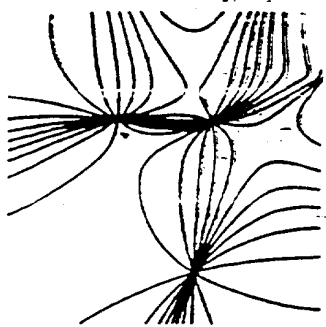
states  $|114\rangle$  and  $|115\rangle$

Circuit

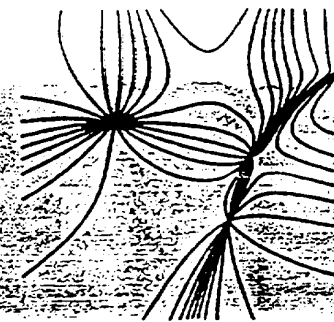




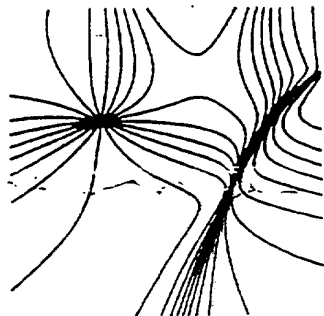
337.5°



343.125°

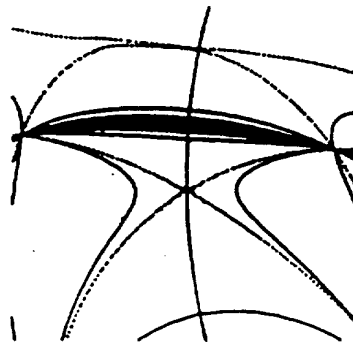


353.25°

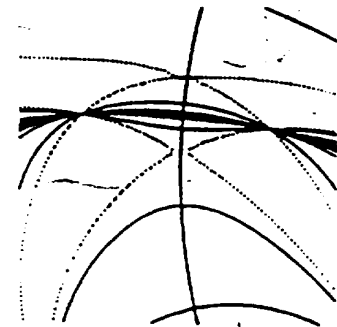


356.25°

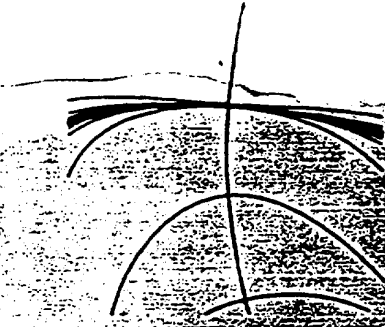
(a)



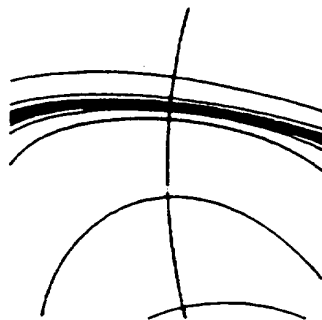
326.25°



328.75°



328.97°



329.0625°

(b)

Figure 10  
Singularities within singularities  
collisions  
dislocation

within singularities  
monopole



## 5. Disclinations in waves (Nye 1983)

In general 3-D spatially varying vector waves, there are singular lines where  $\underline{E}$  is linearly polarized or circularly polarized.

$2\pi$  anholonomy

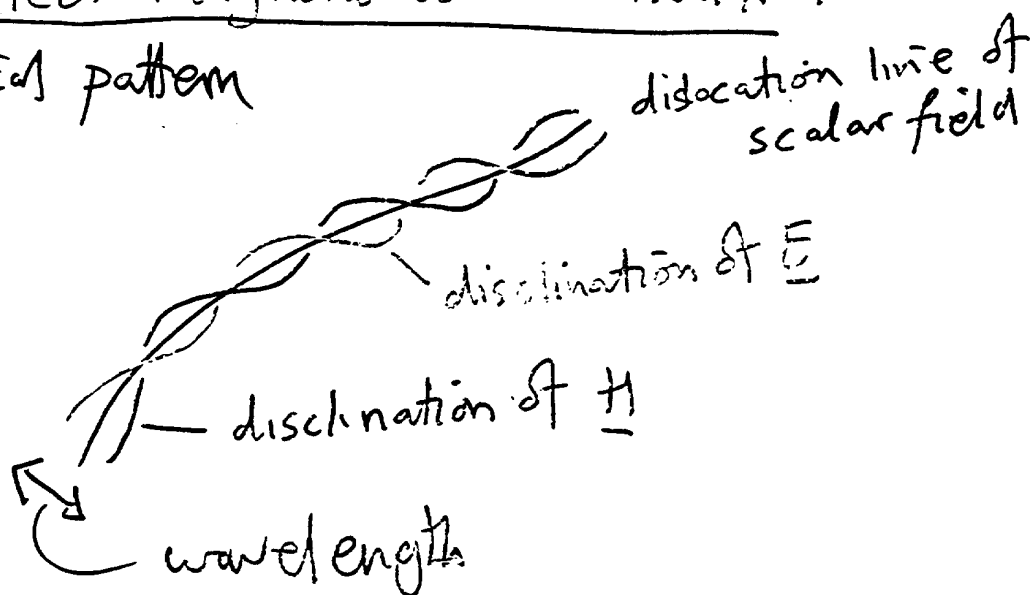
$\pi$  anholonomy

These line singularities are 'a useful way of describing the spatial and temporal geometrical features of electromagnetic fields that would otherwise be far too complicated to visualize. The disclination carries with it a certain local field structure; therefore, a description of the arrangement and motion of the disclinations contains much of the essential geometrical information about the field itself. They constitute elements of structure in the field.'

NPL

### Electromagnetic double helix:

typical pattern



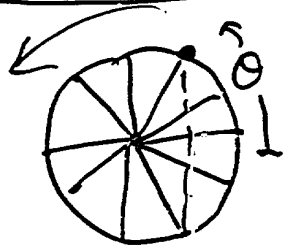
# NEWTONIAN ANHOLONOMY

2.18

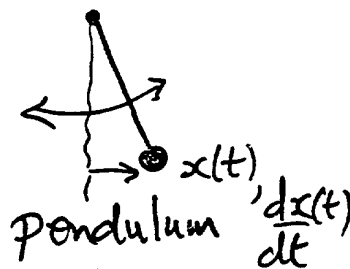
A quantum state with a fixed environment is just a complicated oscillator, and shows phase anholonomy as the environment is altered. In classical mechanics, oscillators abound — wheels, pendulums, in fact any periodic or multiply periodic motion — so these should show anholonomy too.

$$x(t; X) = A(X) \sin \left\{ \omega(X) t \right\}$$

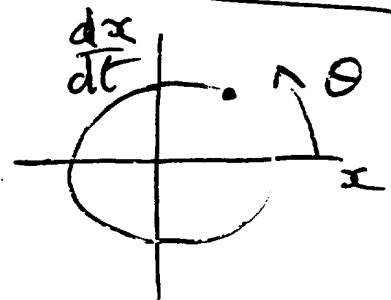
oscillator coordinate  $x(t; X)$     parameters  $X$     amplitude  $A(X)$     frequency  $\omega(X)$     generalized phase angle  $\Theta$



wheel  
→  $x(t)$



pendulum  $x(t)$ ,  $\frac{dx(t)}{dt}$



$\Theta$  is an abstract angle in phase(!) space

$\Theta$  is an angle in space

over time  $T$ ,  $\Theta$  changes by  $\omega(X)T$  for fixed environment  $X$ . When  $X$

is slowly cycled, then, as before,  $\Theta$  acquires not only the expected dynamical change but a geometric one.

$$\Theta_{\text{end}} - \Theta_{\text{start}} = \int_0^T dt \omega(X(t)) + \Delta\Theta(C)$$

dynamical    Hannay's angle

$$\omega = -\frac{d\Theta}{dt}$$

chaos?

# Hannay's angles in classical multiply periodic systems.

Classical Hamiltonian

$$H(\underline{q}, \underline{p}, \underline{X}(H)) \quad D \text{ freedoms}$$

$\swarrow \quad \downarrow \quad \searrow$   
 $q_1, \dots, q_D \quad p_1, \dots, p_D \quad \text{parameters } \{X_1, \dots\}$

For fixed  $\underline{X}$ ,  $H$  is integrable:  $D$  conserved quantities, motion on  $D$ -torus in  $2D$  dim. phase space:

Torus labelled by

$$\underline{I} = \{I_1, \dots, I_D\}$$

$$I_j = \frac{1}{2\pi} \oint_{\gamma_j} p \cdot dq$$

Points on torus labelled by angles  $\underline{\theta} = \{\theta_1, \dots, \theta_D\}$

$1 \leq j \leq D$  :  $D$  actions (conserved)

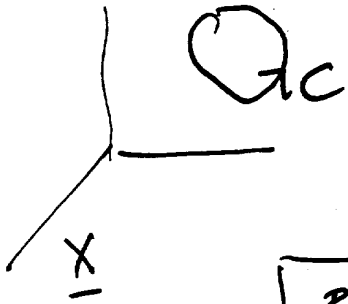
Can express  $H$  in terms of actions

$H = H(\underline{I})$  ; frequencies

$$\omega_j = \frac{\partial H(\underline{I})}{\partial I_j} = \dot{\theta}_j$$

Now let parameters  $\underline{X}$  vary round a cycle  $C$

2.30  
20



At the end, the torus returns ( $\underline{I}$  conserved: classical adiabatic theory)

But  $\underline{\theta}$  does not.

Hannay:

$$\theta_{\text{end}} - \theta_{\text{start}} = \int_0^T dt \omega_j(\underline{X}(t)) + \Delta\theta_j(C)$$

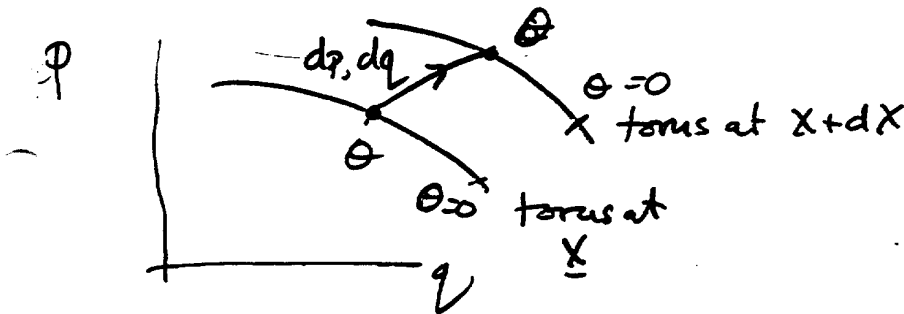
dynamical Hannay

$$\Delta\theta_j = -\frac{\partial}{\partial I_j} \iint_{\partial S=C} \langle d\underline{p} \wedge d\underline{q} \rangle$$

From Hamilton's equations

$$\langle \rangle = \frac{1}{(2\pi)^D} \int_0^{2\pi} d\underline{Q} \int_0^{2\pi} d\underline{P}$$

= torus average



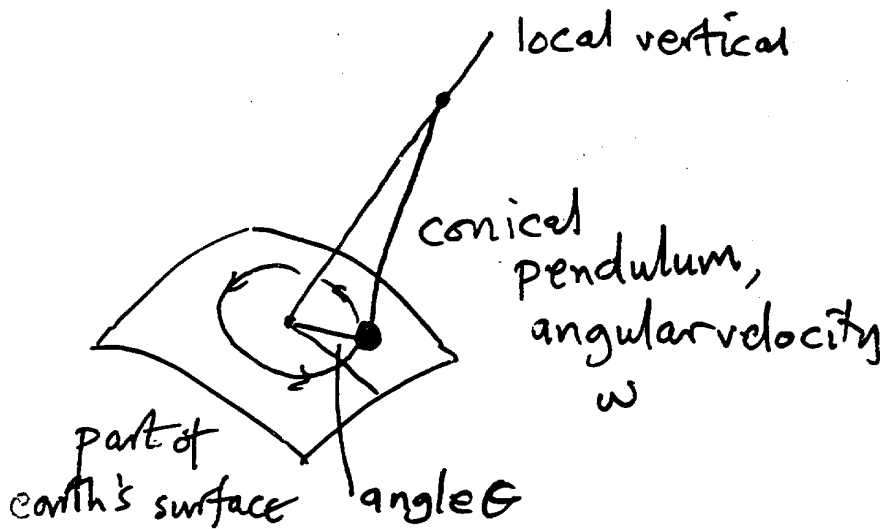
Semiclassical theory : from WKB association  
torus  $\leftrightarrow$  eigenstate

$$\gamma_n(C) \approx \frac{1}{h} \iint_{\partial S=C} \langle d\underline{p} \wedge d\underline{q} \rangle$$

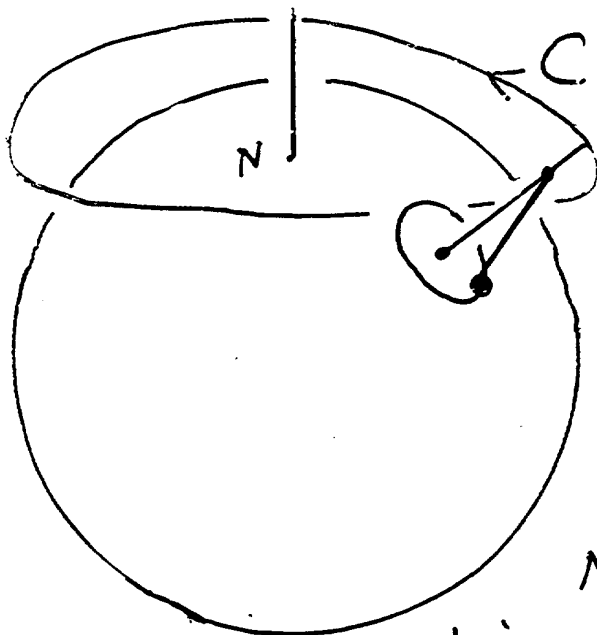
ie  $\Delta\theta_j = -\frac{\partial}{\partial I_j} \gamma_n(C) = -\frac{\partial}{\partial n_j} \gamma_n(C)$

e.g. spin  $\gamma_n(C) = -n/2$ , so  $\Delta\theta = \pi$

# Example : Foucault pendulum



During 24 hours ( $\equiv T$ ), the local vertical (environment) turns :



angle turned through by bob of pendulum is

$$\theta_{\text{end}} - \theta_{\text{start}} = \omega T + \Omega$$

solid angle of  $C$   
- Hannay

Now,  
linear vibration = circular + circular  
=  $\odot + \odot$

So, circular angle shift  
 $\Omega$

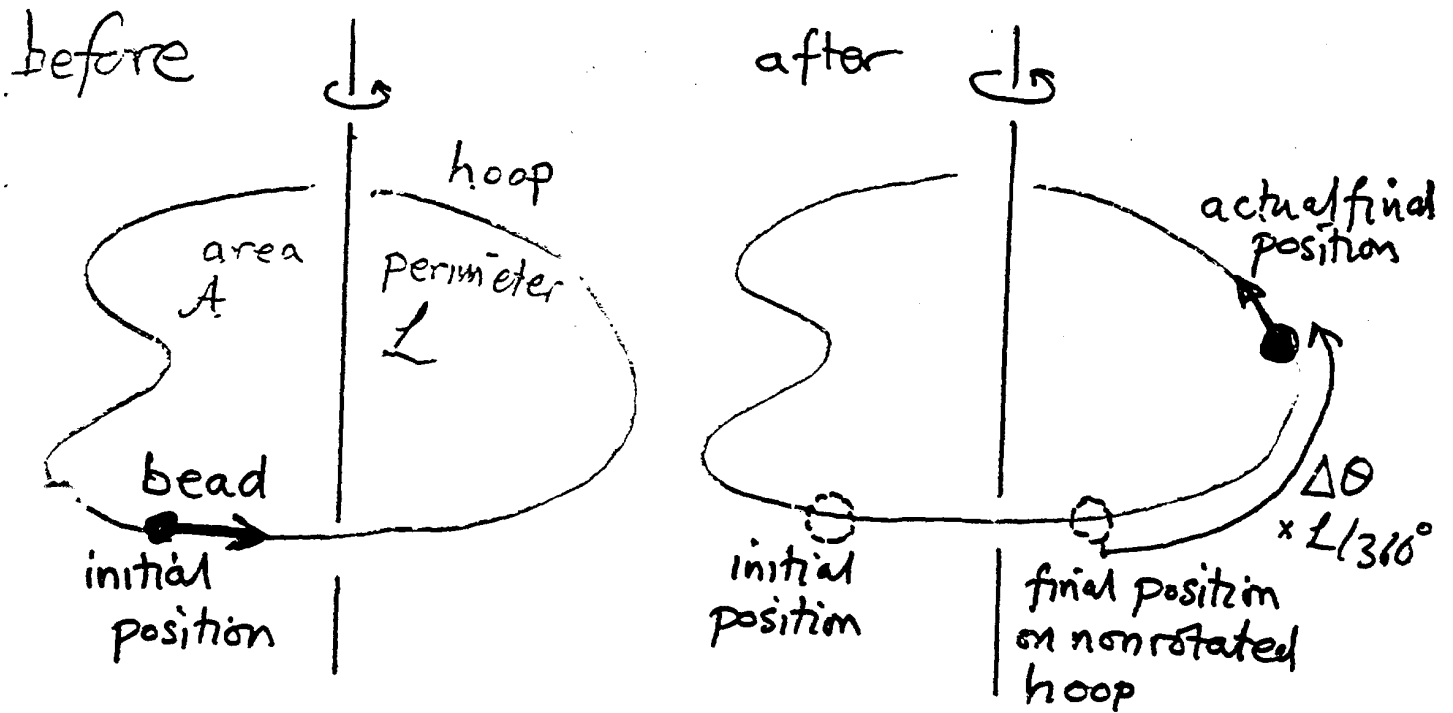
→ rotation of linear vibration  
by  $\Omega = 2\pi(1 - \sin(\text{latitude}))$

→ angular vel. of direction relative to earth's surface  
 $= (\Omega - 2\pi)/T = \frac{2\pi}{T} \sin(\text{latitude})$  (11.7°/hr in 30° lat)

= parallel transport of vibration (again!)

## Example: Rotated rotator (Hannay's hoop)

2.22



Hannay angle:

$$\Delta\theta = 2\pi \left( 1 - \frac{4\pi A}{L^2} \right)$$

purely geometric; zero for circle;  
Euler force.

For chaotic classical dynamics, no tori  
no actions, no Hannay angles.

Does

$$V = \lim \langle dn \cdot dn \rangle$$

$$\left( \gamma_n(c) = \oint_{\partial S=c} V \right)$$

have a classical limit? Yes. Main steps of derivation

$$1) \quad \sum_m |m\rangle \langle m| = 1$$

$$2) \quad \langle m | dn \rangle = \frac{\langle m | d\hat{H} | n \rangle}{E_n - E_m} \quad (\text{twice})$$

$$3) \quad \frac{1}{(E_n - E_m)^2} = -\frac{1}{\hbar^2} \int_0^\infty dt e^{\frac{i\hbar}{\hbar} (E_n - E_m)t}$$

$$4) \quad \begin{array}{cc} \text{quantum commutator} & \text{classical Poisson bracket} \\ \left[ \hat{a}, \hat{b} \right] \rightarrow i\hbar \{a(z), b(z)\} & z = \{q, p\} \end{array}$$

$$5) \quad \begin{array}{l} \langle n | \hat{a} | n \rangle \rightarrow \langle a(z) \rangle \\ = \frac{1}{\partial_E \Omega} \int dz \delta(E - H(z, X)) a(z) \end{array}$$

average over classical  
invariant manifolds  
corresponding to  $|n\rangle$

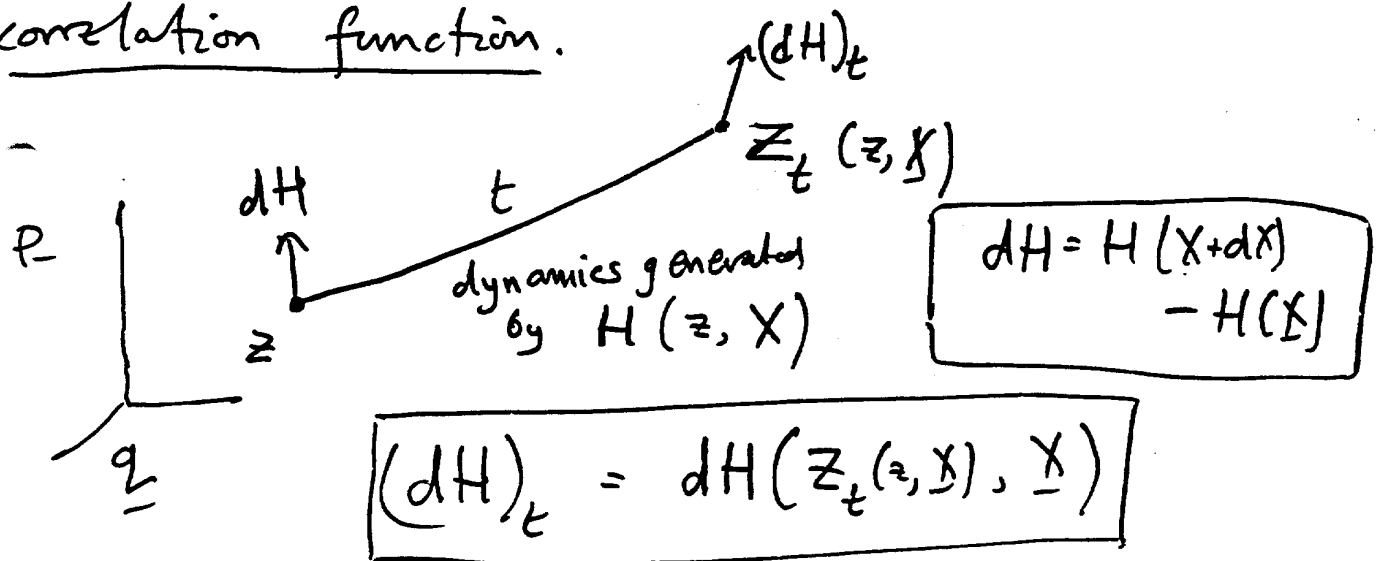
in ergodic case

phase volume of energy surface  $H=E$

Result

$$V(X) \xrightarrow{\hbar \rightarrow 0} -\frac{1}{2\hbar \partial_E \Omega} \partial_E \left[ \partial_E \Omega \int_0^\infty dt \langle (dH)_t \wedge dH \rangle \right]$$

This is the time integral of a classical correlation function.



$V$  involves in an essential way the history of the chaos estimate:



Thus

$$V(X) \approx -\frac{1}{2\hbar \partial_E \Omega} \partial_E \left[ \frac{\partial_E \Omega}{\lambda^2} \langle \{H, dH\} \wedge dH \rangle \right]$$

particles in B fields:  $\langle L_t \wedge L \rangle$



# Lecture 3

(Some geometric phases, Michael Berry Geneva 1993)

## DYNAMICS OF THE ENVIRONMENT

So far, have divided the world into

$$\text{World} = \text{system} + \text{environment}$$

e.g. spin, electron

acting through parameters  $X(t)$

But there is no unilateral action in physics, so the system must react back on the dynamics of the parameters  $X(t)$ , which are themselves subject to quantum mechanics. One reaction effect is on the phase of the quantum state of the environment: roughly, the world's state is single-valued, so the environment must acquire a phase  $-\gamma_n(C)$  to compensate the system's  $+\gamma_n(C)$ .

This affects molecules when divided into:

$$\text{molecule} = \text{electrons} + \text{nuclei}$$

"system" - light,  
swift

"environment" - ponderous,  
slow

Environmental parameters are nuclear position coordinates  $X$ , whose quantization gives the vibration-rotation spectrum of the molecule.

# 6. Molecular electronic degeneracies

3.2

(Longuet-Higgins, Öpik, Pryce, Sack 1958)

Head & Bristol Physics dept

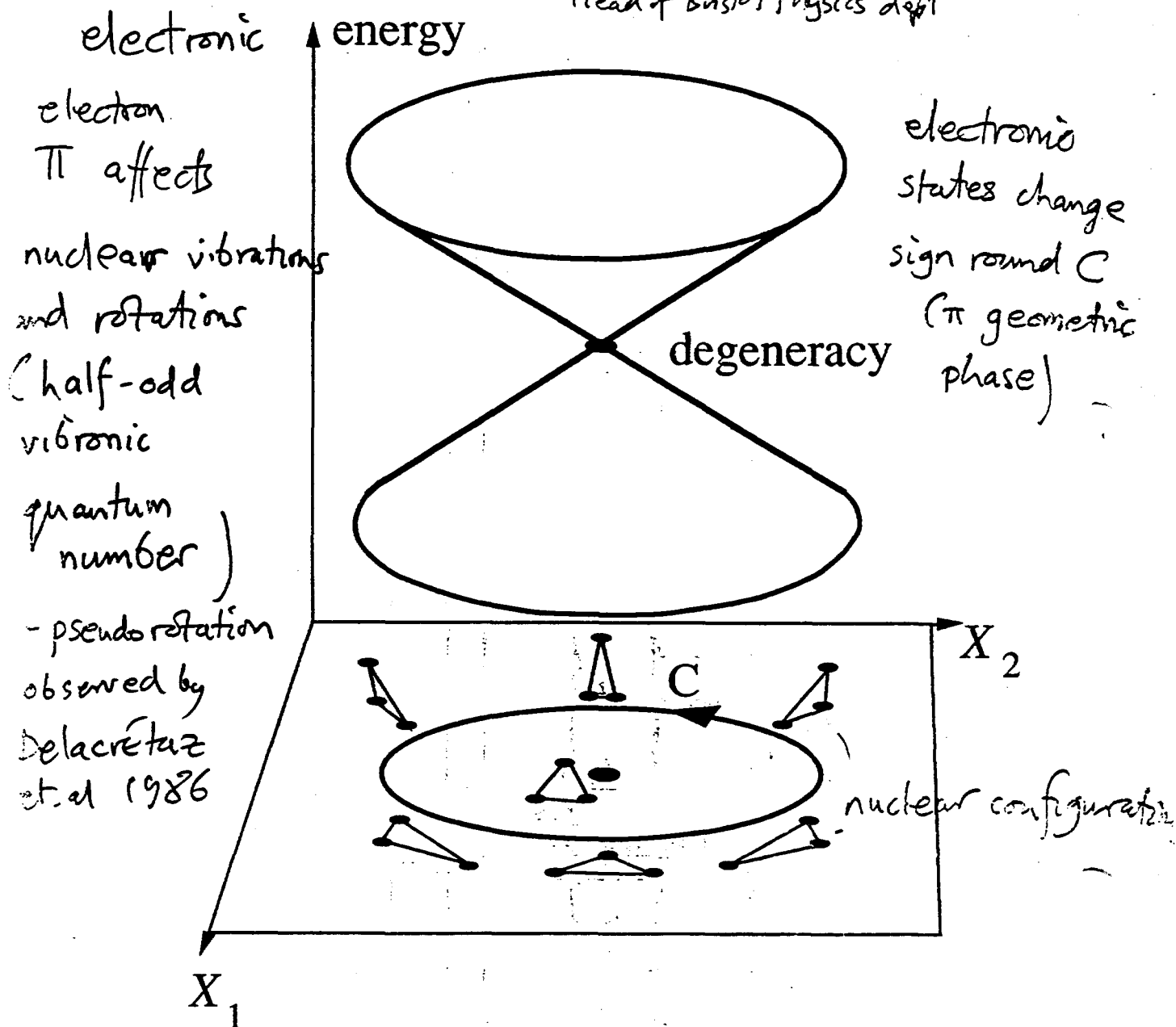


Figure 4. Circuit in shape space  $X$  of nuclei surrounding the equilateral molecule for which there is an electron energy degeneracy (conical intersection)

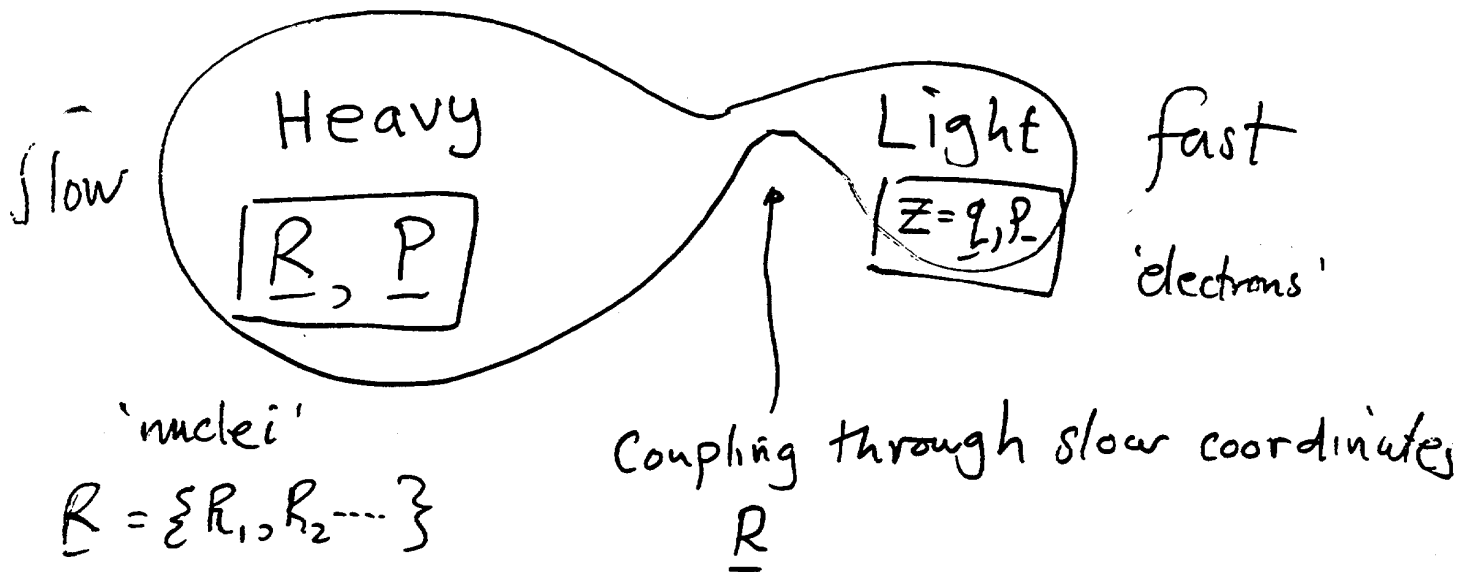
'This half oddness is at first sight strange, but may be understood by noting that [around a circuit] the electronic factor in the wave function will be multiplied by  $-1$ , so that the angular part of the nuclear factor must do likewise if the total wavefunction is to be single-valued.'

(wheel balancing)

Generalization :

## CLASSICAL GEOMETRIC REACTION FORCES

Coupled systems (classical or quantum)



Here, concentrate on slow dynamics.

Simplest approximation (Born-Oppenheimer or Adiabatic averaging) :

Averaged fast energy  $E_{B0}(\underline{R})$  depends on slow coordinates, and its gradient acts as a reaction force on the slow system

$$-\partial_i E_{B0}(\underline{R})$$

$$\partial_i \equiv \frac{\partial}{\partial R_i}$$

Born-Oppenheimer approximation depends on an adiabatic separation of time scales:

Fast motion can be calculated with slow coordinates  $\underline{R}$  acting as parameters, i.e. frozen.

An improved approximation, more consistent with the adiabatic assumption, includes

two extra reaction forces on the slow motion: 'magnetic' and 'electric'

These first appeared in quantum mechanics (modifying vibration-rotation ('slow') energy levels in molecules) but have classical counterparts.

$$H(\underline{R}, \underline{P}, z) = \frac{1}{2} \sum_{i,j} \overset{\text{slow}}{\underset{\text{inverse mass matrix (small)}}{Q_{ij}}} P_i P_j + h(z, \underline{p})$$

fast

Gauge approximation to slow motion based on effective Hamiltonian

$$H_g(\underline{R}, \underline{P}) = \frac{1}{2} \sum_{ij} Q_{ij} (\underline{P}_i - \underline{A}_i(\underline{R})) (\underline{P}_j - \underline{A}_j(\underline{R})) + E_{Bo}(\underline{R}) + \underline{\Phi}(\underline{R})$$

So 3 reaction forces act

Born - Oppenheimer

$$- \partial_i E_{Bo}(\underline{R})$$

Magnetic gauge

$$B_{ij} = \partial_i A_j(\underline{R}) - \partial_j A_i(\underline{R})$$

Electric gauge

$$- \partial_i \underline{\Phi}(\underline{R})$$

Quantum formulae for geometric forces: let fast  $h$  have eigenstates  $|n(\underline{R})\rangle$  and eigenvalues  $E_n(\underline{R})$ . Adiabatic assumption: no transitions, i.e. fast state remains fixed in one of the  $|n\rangle$ 's. Then effective slow Hamiltonian is

$$H_g = \langle n | H | n \rangle = \frac{1}{2} \sum_{ij} Q_{ij} \langle n(\underline{R}) | \underline{P}_i \underline{P}_j | n(\underline{R}) \rangle + E_n(\underline{R})$$

But  $\underline{P}_i = -i\hbar \frac{\partial}{\partial R_i}$ , so

$$\underline{P}_j |n\rangle = -i\hbar \partial_i |n\rangle + |n\rangle \underline{P}_j$$

leading to

$$E_{B0}(\underline{R}) = E_n(\underline{R})$$

$$B_{ij}(\underline{R}) = i\hbar \left( \langle \partial_i n | \partial_j n \rangle - \langle \partial_j n | \partial_i n \rangle \right)$$

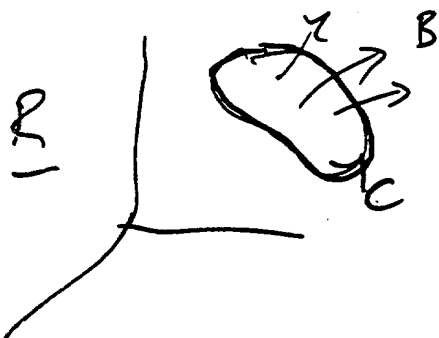
$$2\text{form} : \quad \langle dn | \wedge | dn \rangle$$

$$\Phi(\underline{R}) = \sum \sum Q_{ij} g_{ij}$$

$$g_{ij} = \frac{\hbar^2}{2} \langle \partial_i n | (1 - |n\rangle \langle n|) | \partial_j n \rangle$$

NB:  $B_{ij}$  and  $\Phi$  reflect (in the slow system) geometric aspects of the fast motion:

Geometric phase: in a cycle of  $\underline{R}$ , phase of fast state changes by



$$\gamma_n(C) = i \oint_C \langle dn | \wedge | dn \rangle$$

Fast metric

$$\begin{array}{cc} |n(\underline{R})\rangle & |n(\underline{R}+d\underline{R})\rangle \\ \bullet & \bullet \\ \underline{R} & \underline{R}+d\underline{R} \end{array}$$

$$\begin{aligned} (\text{distance})^2 &\equiv | - \langle n | n + dn \rangle |^2 \\ &= g_{ij} dR_i dR_j \end{aligned}$$

(Provost & Vallee)

Remarkably,  $\hbar$  drops out of the classical limit of these formulae, leaving geometric reaction forces in purely classical dynamics.

Main steps of classical limit: (with J.M. Robbins)

$$1) \sum_m |m\rangle \langle m| = 1$$

$$2) \langle m | \partial_i h | n \rangle = \frac{\langle m | \partial_i h | n \rangle}{E_n - E_m} \quad (\text{twice})$$

$$3) \frac{1}{(E_n - E_m)^2} = -\frac{1}{\hbar^2} \int_0^\infty dt t e^{\frac{i}{\hbar}(E_n - E_m)t}$$

$$4) \begin{array}{l} \text{quantum commutator} \quad \text{classical Poisson bracket} \\ [a, b] \rightarrow i\hbar \{a(z), b(z)\} \end{array}$$

$$5) \langle n | a | n \rangle \rightarrow \langle a(z) \rangle \quad \begin{array}{l} \text{average over} \\ \text{classical invariant} \\ \text{manifold corresponding} \\ \text{to } |n\rangle \end{array}$$

$$= \frac{1}{\int_{\mathcal{R}} dz} \int dz \delta(E - h(z, R)) a(z) \quad \begin{array}{l} \text{phase volume} \\ \text{in ergodic} \\ \text{case} \end{array}$$

These lead to classical formulae that apply also to chaotic fast systems:

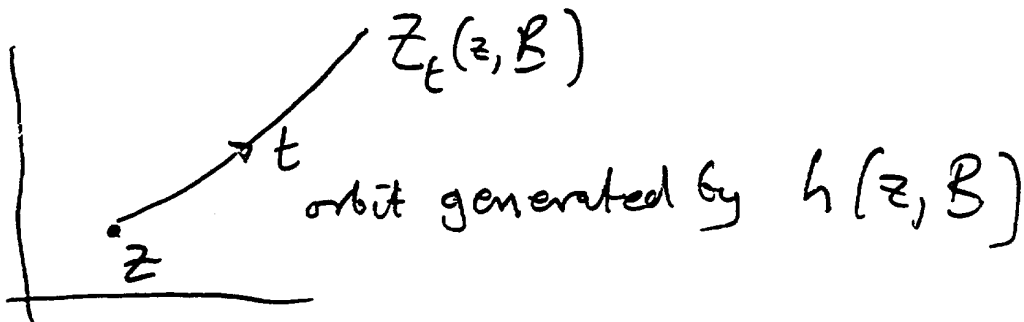


$E_{B_0}(B) = E(B)$ , determined by constancy of adiabatic invariant: phase volume

$$\Omega(E, B) = \int dz \bigoplus_{\text{unit step}} \{E - h(z, B)\}$$

$$B_{ij}(B) = -\frac{1}{2\partial_E \Omega} \partial_E \left[ \partial_E \Omega \int_0^\infty dt \left\langle (\partial_i h)_t \partial_j h - (\partial_j h)_t \partial_i h \right\rangle_E \right]$$

where  $(\partial_i h)_t \equiv \partial_i h(Z_t(z, B), B)$



$$g_{ij}(B) = -\frac{1}{2} \int_0^\infty dt \, t \left\langle (\partial_i h)_t - \partial_i E \right\rangle \left\langle (\partial_j h)_t - \partial_j E \right\rangle_E$$

fluctuation in  $\partial_i h$  at  $t$

These involve correlations

$$C_{ij}(t) \equiv \left\langle (\partial_i h - \partial_i E)_t (\partial_j h - \partial_j E) \right\rangle_E$$

Connection with linear response theory:

Can derive geometric forces from classical mechanics, using shift of distribution of fast variables away from microcanonical.

In the chaotic case, get, in addition,

**{FRICTION}** - dissipation of slow motion by energy absorption by fast motion. Velocity dependent forces

$$\ddot{R}_i = -k_{ij} \dot{R}_j$$

antisymmetric part = magnetic force  
symmetric part is friction

$$\left[ \frac{1}{2\partial_E \Omega} \partial_E \left[ \partial_E \Omega \int_0^\infty dt [C_{ij}(t) + S_{ij}(t)] \right] \right]$$

N.B. Rate of energy absorption by fast chaos is

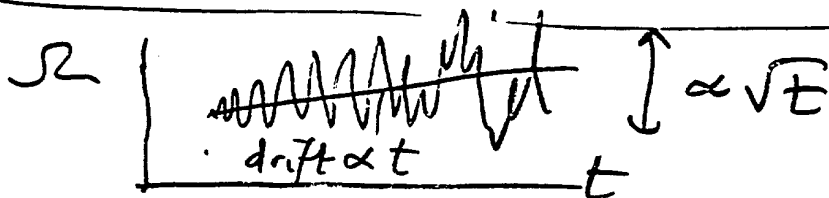
$$\dot{E}_a = \dot{R}_i k_{ij} \dot{R}_j$$

Drift in adiabatic invariant

$$\dot{\Omega} = \partial_E \Omega \dot{R}_i k_{ij} \dot{R}_j$$

- superimposed on fluctuations

$$\left( \Delta \Omega \right)_t^2 \xrightarrow{t \rightarrow \infty} \frac{(\partial_E \Omega)^2}{2} \dot{R}_i \dot{R}_j \left[ t \int_0^\infty d\tau C_{ij}(\tau) - \int_0^\infty d\tau \tau C_{ij}(\tau) \right]$$



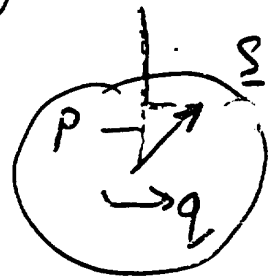
# Spin model (with J. M. Robbins)

3.10

$$H = \frac{\underline{P}^2}{2} + \underline{R} \cdot \underline{S}$$

Heavy particle in 3D, coupled with spin  $\underline{S}$  via position  $\underline{R}$  (sphere of monopoles)

$\underline{S}$  is fast system (1 freedom)



Adiabatic regime

$$R = |\underline{R}| \text{ large}$$

(from eliminating mass)

Exact equations of motion (4 freedoms, 8D phase space)

$$\text{fast: } \dot{\underline{S}} = \underline{R} \wedge \underline{S}$$

$\underline{S}$  precesses about instantaneous  $\underline{R}$

$$\text{slow } \underline{\dot{R}} = \underline{V} = \underline{P}, \quad \underline{\ddot{R}} = \underline{\dot{P}} = -\underline{S} \quad \underline{S} \text{ forces } \underline{R}$$

Motion probably nonintegrable, but can eliminate fast motion (unusual!) exactly, using conservation of total angular momentum:

$$\underline{J} = \underline{R} \wedge \underline{V} + \underline{S} = \text{constant}$$

Thus

$$\boxed{\dot{\underline{V}} = \underline{R} \wedge \underline{V} - \underline{J}} \quad \left( \begin{array}{l} \text{measure-preserving} \\ \text{but not Hamiltonian} \\ \nabla \cdot \underline{R} \neq 0 \end{array} \right)$$

(charged particle in monopole, plus 'gravity')

conserved:

$$\boxed{E = \frac{1}{2} V^2 + \underline{J} \cdot \underline{R}} \quad (\text{energy})$$

$$\boxed{|\underline{S}| = |\underline{J} - \underline{R} \wedge \underline{V}|} \quad (\text{magnitude of acceleration})$$

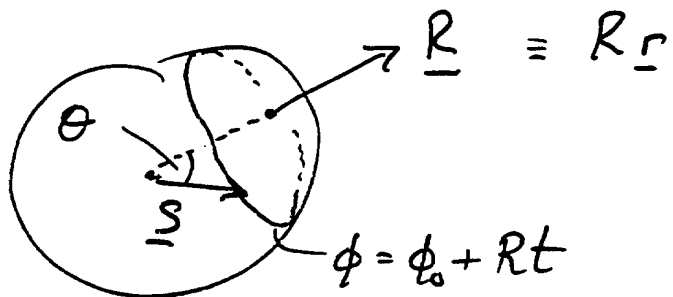
Geometric forces

Fast motion generated by

$$h = \underline{R} \cdot \underline{S}, \text{ i.e. } \dot{\underline{S}} = \underline{R} \wedge \underline{S}, \underline{R} \text{ fixed}$$

need  $\nabla_{\underline{R}} h = \underline{S}$

motion integrable and ergodic

Invariant manifold : latitude circle  $\theta$ 

Adiabatic invariant

$$\boxed{I = \underline{S} \cdot \underline{r} = S \cos \theta} \quad \left( = \frac{1}{2\pi} \mathcal{Q} = \frac{1}{2\pi} \oint P dq \right)$$

Thus  $E_{B0} = I R$

$$\underline{B} = -\frac{1}{2} \partial_E \int_0^\infty dt \langle \underline{S}_t \wedge \underline{S} \rangle_E$$

$$\underline{\Phi} = -\frac{1}{2} \int_0^\infty dt \langle (\underline{J}_t - I \underline{r}) \cdot (\underline{S} - I \underline{r}) \rangle_E$$

|          |                                                            |
|----------|------------------------------------------------------------|
| Magnetic | $\underline{B} = -\underline{I} \frac{\underline{r}}{R^2}$ |
| electric | $\underline{\Phi} = \frac{S^2 - \underline{I}^2}{2R^2}$    |

monopole

inverse-cube  
repulsive force

Thus adiabatic Theory predicts acceleration

|                       |     |                                |                                                                  |                                                       |                     |
|-----------------------|-----|--------------------------------|------------------------------------------------------------------|-------------------------------------------------------|---------------------|
| $\underline{\dot{V}}$ | $=$ | $-\underline{I} \underline{r}$ | $- \underline{I} \frac{\underline{V} \wedge \underline{r}}{R^2}$ | $+ \frac{(S^2 - \underline{I}^2) \underline{r}}{R^3}$ | Aharonov<br>& Stern |
|                       |     | Born-Oppenheimer               | magnetic                                                         | electric                                              |                     |

Study the geometric forces in isolation.

Magnetic only

Eliminate Born-Oppenheimer and electric (both radial) by confining motion to a sphere  $R = \text{constant}$ . Implement this constraint with

$$\underline{P}^2 \rightarrow \underline{P}^2 - (\underline{P} \cdot \underline{r})^2$$

Then

$$\underline{V} = \underline{P} - \underline{P} \cdot \underline{r} \underline{r} \text{ i.e. } \underline{V} \cdot \underline{r} = 0$$

Exact equations of motion for slow variables

$$\dot{\underline{V}} = \underline{R} \wedge \underline{V} - \underline{J} + \underline{J} \cdot \underline{r} \frac{\underline{r}}{R} - \frac{V^2}{R} \frac{\underline{r}}{R}$$

(conserves  $R$ ,  $E$ ,  $S = |\underline{J} - \underline{R} \wedge \underline{V}|$ ) seek  $\underline{r} = (\theta, \phi)$   
about  $\underline{J} = J_z \hat{z}$

This motion is integrable:

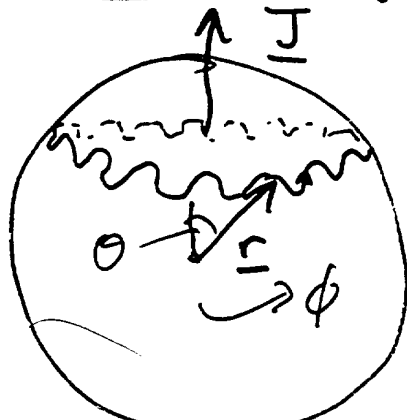
$$K \equiv E + \frac{1}{2R^2} (J^2 - S^2) = \underline{R} \cdot \underline{J} + \frac{(\underline{r} \wedge \underline{V}) \cdot \underline{J}}{R}$$

conserved

$$\therefore 2(E - K) + \frac{|\underline{J} \wedge \underline{r}|^2}{R^2} = \left( \underline{V} - \frac{\underline{J} \wedge \underline{r}}{R} \right)^2$$

$$\begin{aligned} \frac{2(E - K)}{R^2} + \frac{J^2 \sin^2 \theta}{R^4} &= \dot{\theta}^2 + \sin^2 \theta \left( \dot{\phi} - \frac{J}{R^2} \right)^2 \\ K &= J (\sin^2 \theta \dot{\phi} + R \cos \theta) \end{aligned}$$

Motion is precession about  $\underline{J}$  ( $\dot{\phi}$ )  
with nutation ( $\theta$  oscillations)



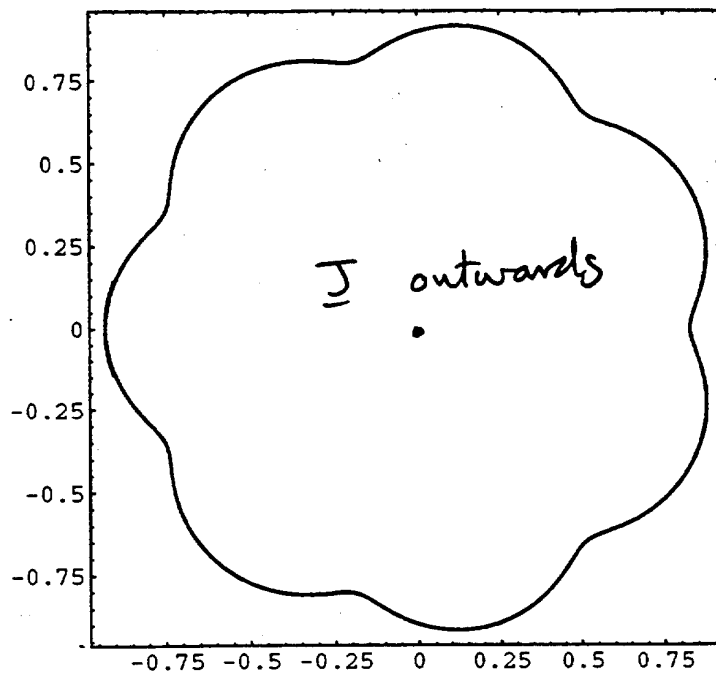
Adiabatic limit

$R$  large:  $\dot{\theta} \rightarrow 0$   
(no nutation)

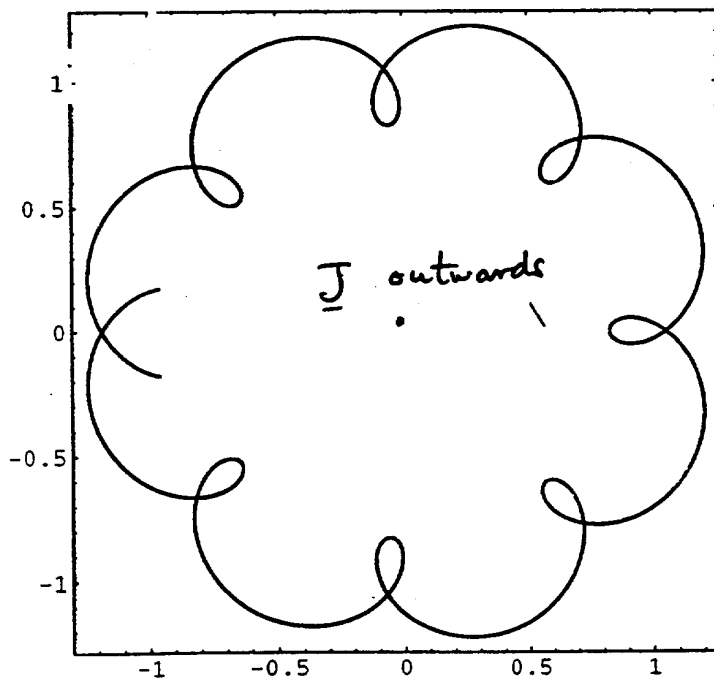
$$\dot{\phi} = \frac{J}{R^2} \rightarrow \underline{V} = R \sin \theta \dot{\phi} = \frac{J \tan \theta}{R}$$

$$(I = \underline{r} \cdot \underline{R} = \underline{J} \cdot \underline{R} = JR \cos \theta)$$

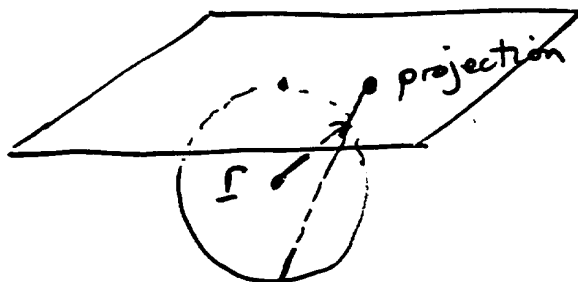
# Stereographic projections from $S^2$ to $\mathbb{R}^2$ sphere <sup>3.14</sup>

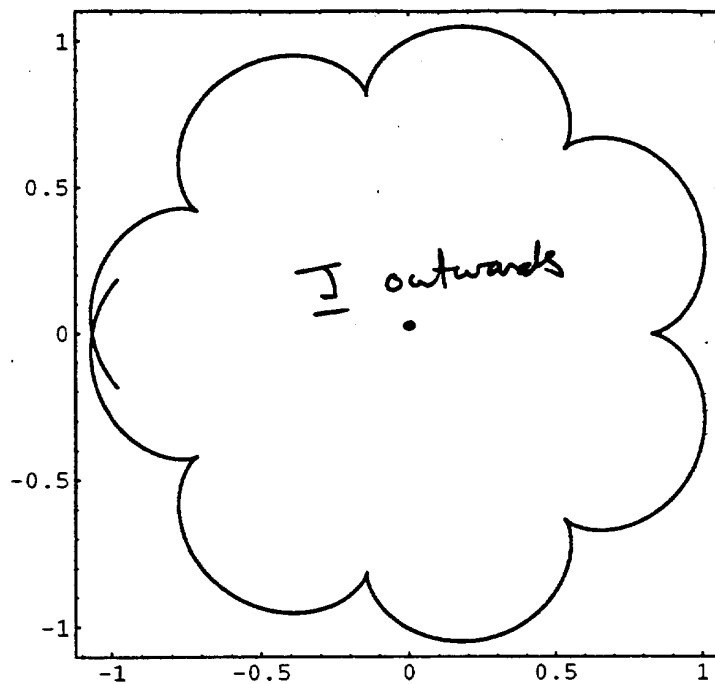


$$R=3 \quad J=3$$

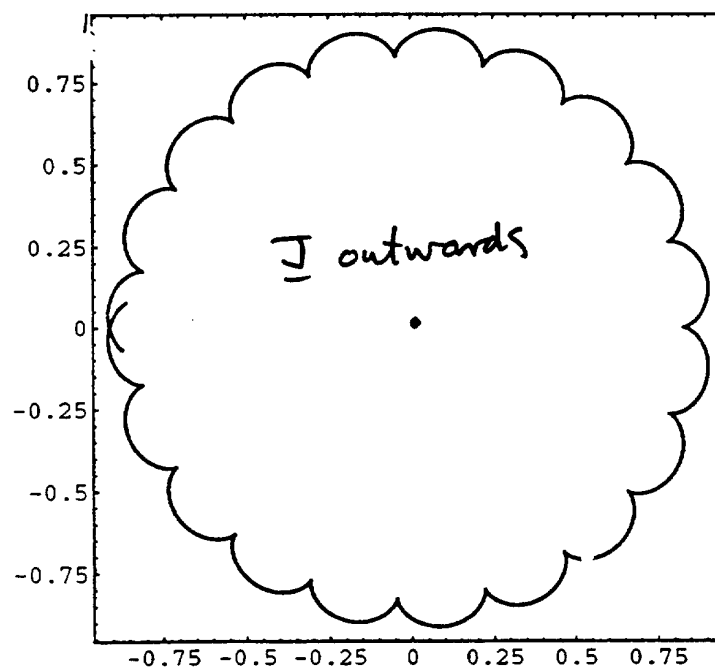


$$R=3 \quad J=3$$





$$R=3 \quad J=3$$



$$R=4 \quad J=3$$

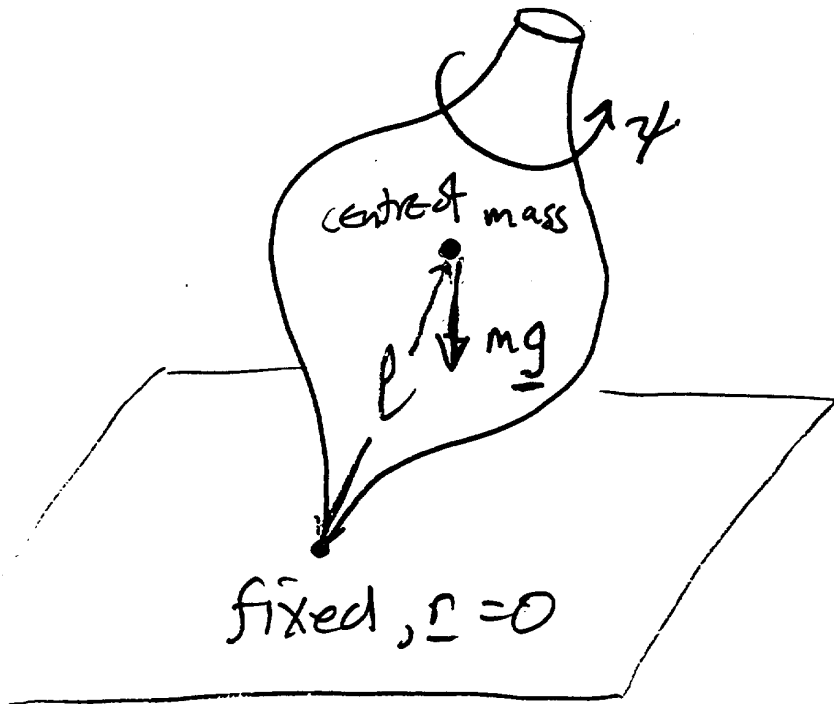
Adiabatically, amplitude of nutation  
diminishes



Analogy with heavy symmetrical (Lagrange) top

$$\uparrow \underline{r} = (\theta, \phi)$$

Moments of inertia  
 $A, B, B$



Angular velocity

$$\underline{\Omega} = (\dot{\psi} + \dot{\phi} \cos \theta) \underline{r} + \underline{r} \wedge \dot{\underline{r}}$$

momentum  $\underline{L} = A(\dot{\psi} + \dot{\phi} \cos \theta) \underline{r} + B \underline{r} \wedge \dot{\underline{r}}$

$$L_{\psi} = \underline{L} \cdot \underline{r} = A(\dot{\psi} + \dot{\phi} \cos \theta) \quad \text{conserved}$$

$$L_{\phi} = \underline{L} \cdot \underline{z} = L_{\psi} \underline{r} \cdot \underline{e}_z + B \underline{r} \wedge \dot{\underline{r}} \cdot \underline{e}_z \quad \text{conserved}$$

torque equation  $\dot{\underline{L}} = -mg\ell \underline{r} \wedge \underline{z}$

gives

$$\ddot{\underline{r}} = \frac{L_{\psi}}{B} \underline{r} \wedge \dot{\underline{r}} - \frac{mg\ell}{B} (\underline{z} - \underline{z} \cdot \underline{r} \underline{r}) - |\dot{\underline{r}}|^2 \underline{r}$$

Identical to equations of motion of spin model (constrained to a sphere) if

$$\boxed{R = \frac{L_y}{B} \quad \underline{J} = \frac{mgl L_y}{B^2} \underline{z}}$$

$\therefore$  Adiabatic regime  $\boxed{R \text{ large} \rightarrow L_y \text{ large} = \text{fast top}}$   
 $(K = \frac{J L_y}{B})$

Puzzle: no obvious top analogy for spin

$$\boxed{\underline{S} = \frac{L_y}{B^2} [mgl \underline{z} - L_y \underline{r}_\perp (B \wedge \underline{r})]}$$

Geometric force approximation:

$$\boxed{\underline{\dot{V}} = -I \frac{\underline{V} \wedge \underline{r}}{R^2} - \frac{V^2}{R} \underline{r}} \quad \text{monopole}$$

conserves  $R$ ,  $E = \frac{1}{2} V^2$  and

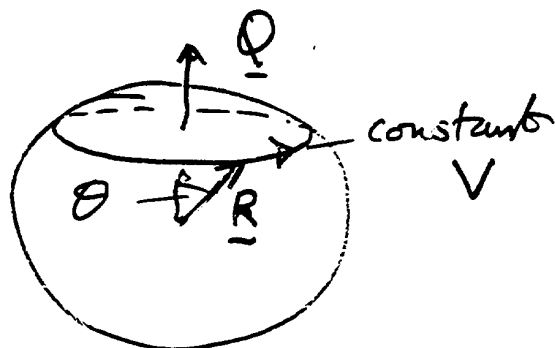
$$\boxed{\underline{\Phi} = R \wedge \underline{V} + I \underline{r}}$$

Hence  $\underline{r} \cdot \underline{\Phi}$  constant  $\rightarrow$

speed given by

$$\tan \theta = \frac{|R \wedge V|}{I}$$

$$\rightarrow \boxed{V = \frac{I \tan \theta}{R}}$$



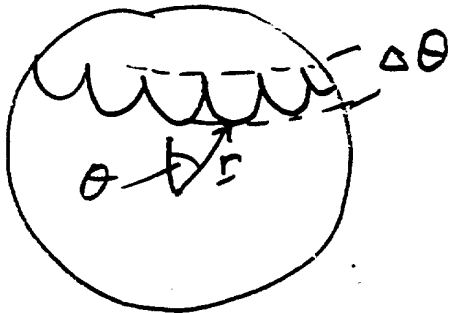
\* Thus, magnetic gauge force describes exactly the adiabatic precession.

Born-Oppenheimer fails completely, predicting  $\underline{B}$  at rest.

Accuracy of conservation of adiabatic invariant

$$I = J \cos \theta$$

$$\boxed{\frac{\Delta I}{I} = \frac{\Delta \cos \theta}{\cos \theta} \rightarrow \frac{+2J \sin^2 \theta_0}{R^3 \cos \theta_0}} \quad \text{as } R \rightarrow \infty$$



So, precession of a top, with the axis (slow) forced by reaction from the spin (fast), is 'caused' by the monopole at the point that is held fixed.

## Electric only

Eliminate Born-Oppenheimer and magnetic by choosing  $\underline{I} = 0$

In particular, choose



$$\boxed{\underline{J} = 0 \rightarrow \underline{I} = \underline{S} \cdot \underline{r} = \underline{J} \cdot \underline{r} = 0 \text{ always}}$$

Exact motion is

$$\boxed{\dot{\underline{V}} = \underline{R} \wedge \underline{V}}$$

('spin'  $\underline{V}$ , driven by  $\underline{R}$ , and coupled by  $\dot{\underline{R}} = \underline{V}$ , or charged particle in uniform sphere of monopoles)

This can be solved exactly.

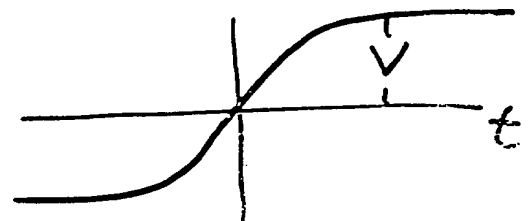
Speed  $\boxed{V = \text{constant.}}$

$$\frac{1}{2}(\ddot{R}^2) = \underline{R} \cdot \dot{\underline{V}} = V^2 \rightarrow \boxed{R^2(t) = R_0^2 + V^2 t^2}$$

radial repulsion.

'Adiabatic invariant' for 'spin'  $\underline{V}$  is not constant:

$$\boxed{\underline{r} \cdot \underline{V} = \frac{\underline{R} \cdot \underline{V}}{R} = \frac{Vt}{\sqrt{t^2 + R_0^2}/\sqrt{2}}}$$



So, asymptotically, orbits recede along radial

Can set  $V=1$ , and parameterize orbit shapes just with  $R_0$

Count of variables: 6 quantities to describe an orbit  
 eliminate: 1 by choosing  $t=0$  at closest approach  
 3 by rigid rotation of orbit about  $\underline{R}=0$   
 1 by scaling:  $\alpha \underline{R}(t)$  is an orbit if  $\underline{R}(t)$  is  
 $\underline{5} \rightarrow 1 \text{ left} = R_0$

Thus time  $t$  = arc length along orbit, so

$\underline{V}(t)$  is unit tangent. Orbit as space curve:

curvature

$$\boxed{\kappa = \sqrt{\underline{\dot{V}} \cdot \underline{\dot{V}}} = \text{const} = R_0}$$

Normal and binormal

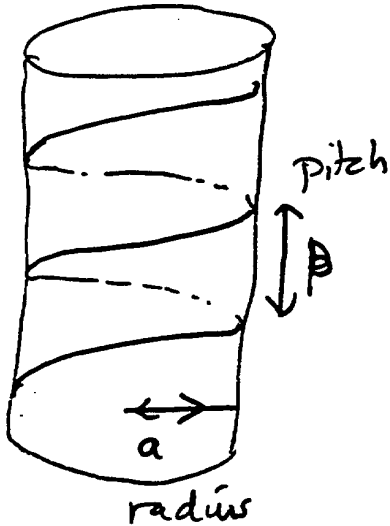
$$\underline{n} = \frac{\underline{R} \wedge \underline{V}}{R_0} \quad \underline{b} = \frac{\underline{R} - t \underline{V}}{R_0}$$

torsion

$$\boxed{\tau = -\underline{\dot{b}} \cdot \underline{n} = t}$$

What curves have constant curvature  $R_0$  and changing torsion  $t$ ?

Preliminary: uniform helix



$$K = \frac{4\pi^2 a}{P^2 + 4\pi^2 a^2}$$

$$\tau = \frac{2\pi P}{P^2 + 4\pi^2 a^2}$$

$$\begin{aligned} a &= \frac{R_0}{R_0^2 + t^2} \rightarrow \frac{R_0}{t^2} \\ P &= \frac{2\pi t}{R_0^2 + t^2} \rightarrow \frac{2\pi}{t} \end{aligned}$$

So, orbits coil round asymptotes with shrinking radius and pitch



Winding reverses at  $t=0$  (zero torsion),  
so predict curly antelope horns for  $\underline{R}(t)$

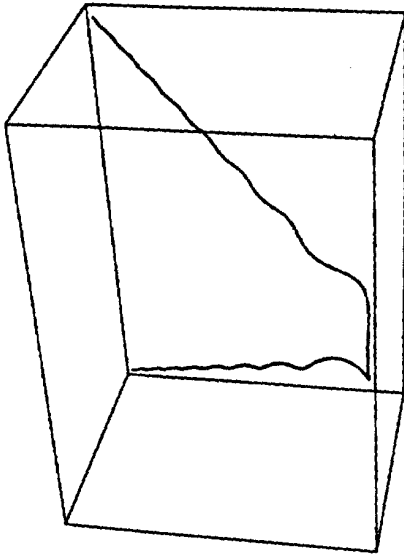
and for  $\underline{V}$  on its unit sphere predict

a curve whose speed is the curvature  $R_0$  of  $\underline{R}(t)$   
and whose curvature is the torsion  $t$  of  $\underline{R}(t)$

i.e. an asymptotically infinitely coiling S

Antelope horns: numerical solution  
of  $\ddot{\underline{R}} = \underline{R}_\perp \ddot{\underline{R}}$

A=1

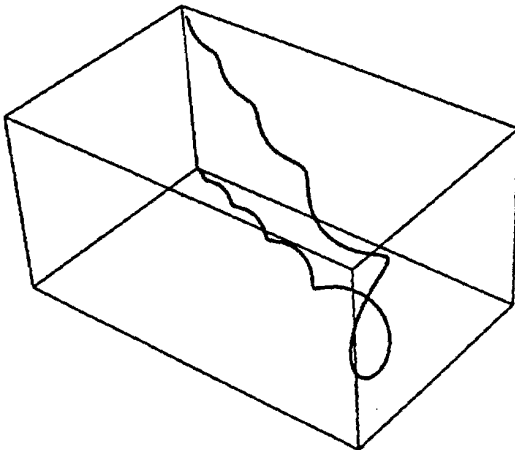
 $R_0 = 1$ 

$$V = |\dot{\underline{R}}| = 1$$

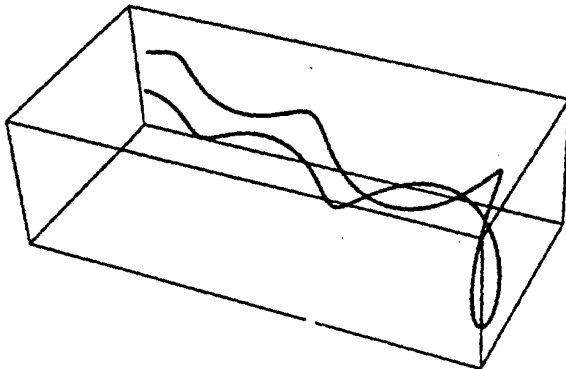
closest approach

$$R = R_0$$

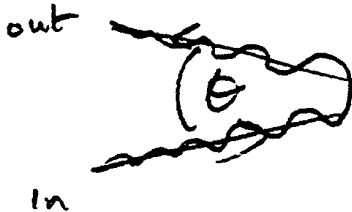
A=1.5

 $R_0 = 1.5$ 

A=2

 $R_0 = 2$ 

Horns close up as  $R_0$  increases

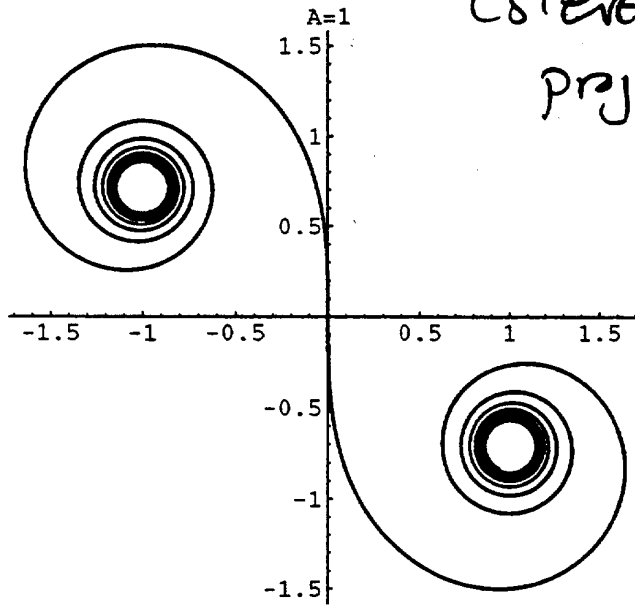


seek opening angle  
 $\Theta(R_0)$

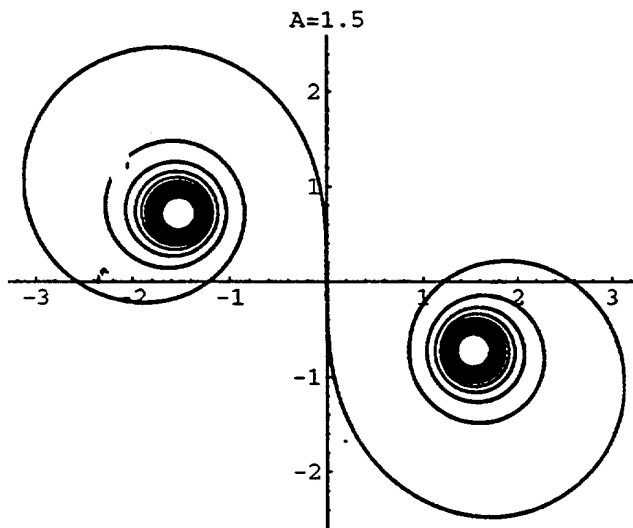
Track of velocity  $\underline{V}$  on its unit sphere

3.23

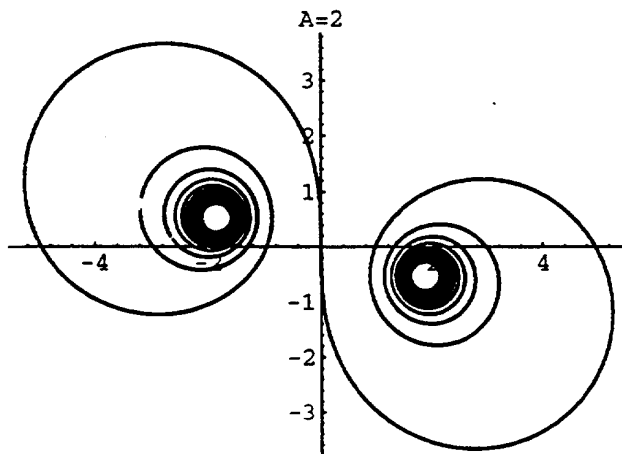
(stereographic S pole projection)



$$R_0 = 1$$



$$R_0 = 1.5$$



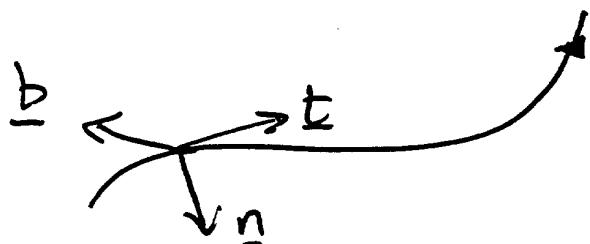
$$R_0 = 2$$

As  $R_0$  increases, velocity asymptotes become antipodal



Intermezzo Constructing the curve  
 with given curvature  $\kappa(s)$  functions of  
 torsion  $\tau(s)$  arc length  $s$

Frenet equations



$$\begin{aligned}\underline{t}' &= \kappa \underline{n} \\ \underline{n}' &= -\kappa \underline{t} + \tau \underline{b} \\ \underline{b}' &= -\tau \underline{n}\end{aligned}$$

9) linear equations. Reduce to 2 by mapping onto evolution of a quantum spin- $\frac{1}{2}$  particle in a 'magnetic field'

$$\underline{B}(s) = (\kappa(s), 0, -\tau(s))$$

Schrödinger equation

$$i|\psi\rangle' = \underline{\sigma} \cdot \underline{B}(s)|\psi\rangle$$

$$|\psi\rangle = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

$$\underline{\sigma} = \frac{1}{2} \left[ \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_x, \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_y, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_z \right] = \text{Pauli matrices}$$

take any 3 solutions  $|\psi_x\rangle, |\psi_y\rangle, |\psi_z\rangle$

with  $\langle \psi_i | \psi_j \rangle = \delta_{ij}$  orthogonal at  $t=0$ .

Then

$$\underline{t} = 2 \left[ \langle \psi_x | \underline{\sigma}_z | \psi_x \rangle, \langle \psi_y | \underline{\sigma}_z | \psi_y \rangle, \langle \psi_z | \underline{\sigma}_z | \psi_z \rangle \right]$$

given  $\underline{t}(s)$ , the curve is  $\underline{R}(s) = \int_0^s \underline{t}(s) ds$

In the spin model  $\dot{\underline{V}} = \underline{R} \wedge \underline{V}$

The 'quantum Hamiltonian' is

$$\underline{\sigma} \cdot \underline{B}(t) = \frac{1}{2} \begin{pmatrix} -t & R_0 \\ R_0 & t \end{pmatrix}$$

defining the  
Landau-Zener  
problem of  
quantum transition  
theory

Thus the velocity (unit tangent) is

$$\underline{V}(t) = \left\{ R_0 \operatorname{Im} y_1 y_2^*, R_0 \operatorname{Re} y_1 y_2^*, |y_1|^2 - \frac{1}{4} R_0^2 |y_2|^2 \right\}$$

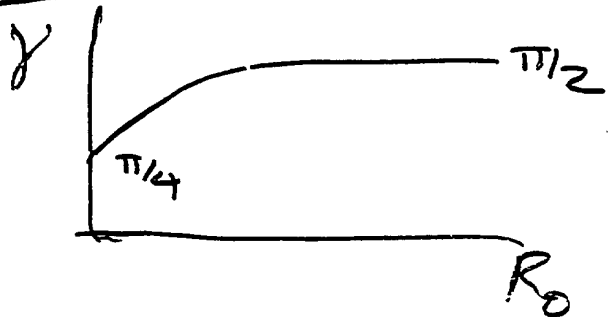
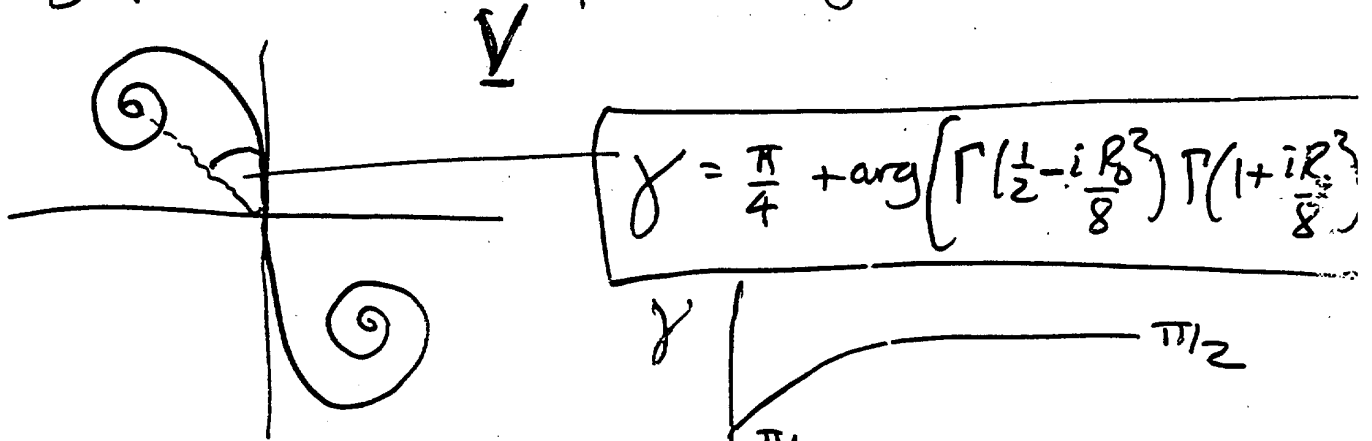
where  $y_1, y_2$  are the even and odd parabolic cylinder (Kummer, confluent hypergeometric) functions

$$y_1(t) = e^{-\frac{it^2}{4}} M\left(\frac{1}{8}iR_0^2, \frac{1}{2}, \frac{1}{2}it^2\right)$$

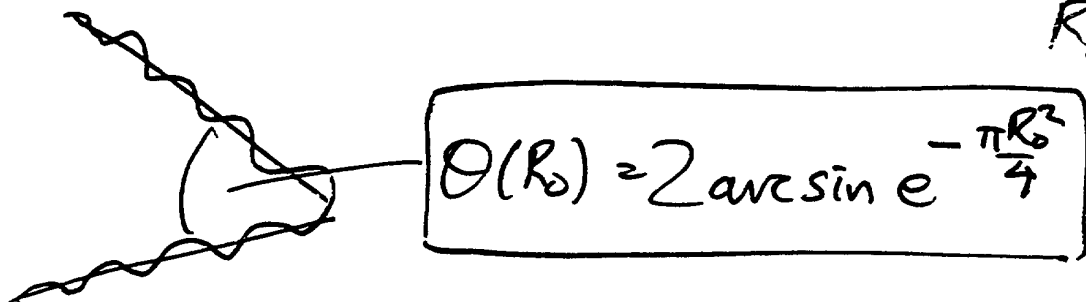
$$y_2(t) = t e^{-\frac{it^2}{4}} M\left(\frac{1}{8}iR_0^2 + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}it^2\right)$$

Asymptotes of these functions gives

3.26



and



## Geometric force approximation

For large  $R_0$ , particle spirals in and back out along same direction, so 'guiding centre' motion is purely radial.

Gauge theory (electric force only)  $I=0$  so

$$\dot{\underline{V}} = \frac{S^2}{R^3} \underline{r}, \quad S = |\underline{R}(t) \wedge \underline{V}(t)| = R_0$$

solution

$$\underline{R}(t) = \sqrt{R_0^2 + t^2} \underline{z}$$

So electric gauge force describes exactly the repulsion of the guiding centre \*

Born-Oppenheimer fails completely.

# Conclusions

1. Geometric reaction forces of electric and magnetic type give a clear picture of average slow motion beyond Born-Oppenheimer, and can capture the main features of this motion with little calculation.
2. New general formulae, giving  $B_{ij}$  and  $\Phi$  in terms of fast orbits with fixed  $\underline{R}$ , are easy to use and apply also to chaotic fast-motion.
3. The particular model  $\frac{P_z^2}{2} + \underline{S} \cdot \underline{R}$  much richer than the two special cases described, and includes chaos in exact slow motion, but geometric forces still produce distinctive effects (e.g. swerving out of plane) observed in numerical computations.
4. For chaotic fast motion, the magnetic force is the neglected antisymmetric cousin of friction.

# MORE CLASSICAL ANHOLONOMY

## SWIMMING BUGS (Shapere and Wilczek)

Swimming = cyclic alteration of a body's shape producing net translation through a fluid.



The shape variables return; the position variables do not. Therefore swimming is anholonomy.

For small creatures (insects, bacteria) the translation vector depends only on the sequence of body shapes, and not on how fast the movements are made, because the fluid reaction is dominated by viscous (rather than inertial) forces. So reciprocating motions get nowhere.

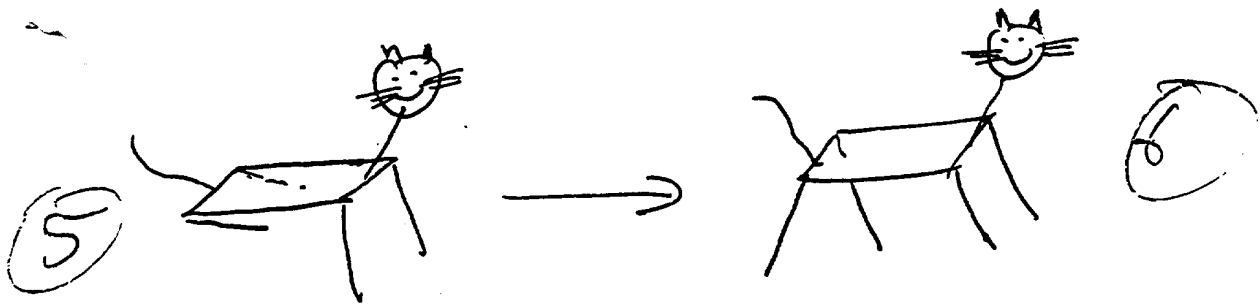
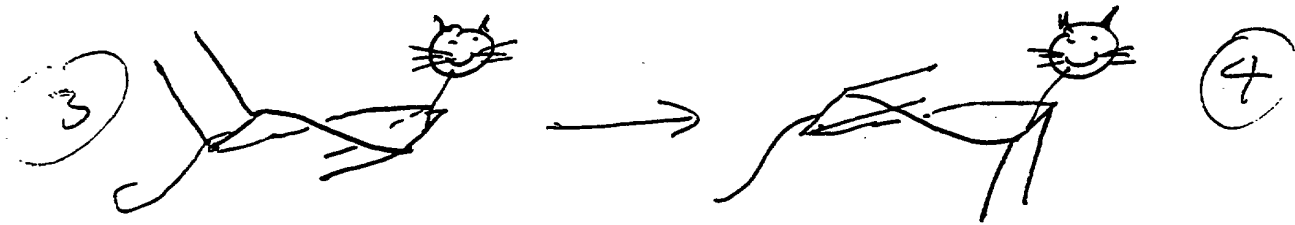
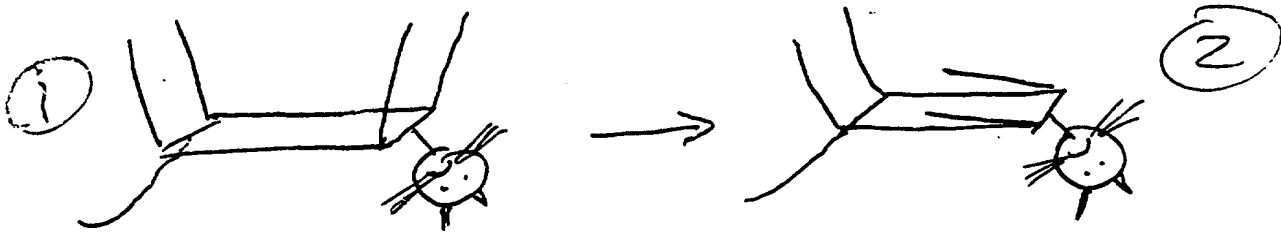
The body makes a circuit  $C$  in the abstract space of shapes. Wilczek et. al show:

anholonomy (= translation vector)

= flux of something (complicated) through  $C$ .

# CATS IN SPACE (Shapere and Wilczek)

With nothing to push against, the cat's angular momentum remains zero. Nevertheless it can rotate, by changing its shape:



The shape variables return, but the orientation variables do not, so the cat's turn is anholonomy.

A formula for the net rotation of the cat: <sup>3.30</sup>

$R_{ij}$  = rotation matrix turning initial into final object (with the same shape).

$$= P \exp \left[ \oint A_{ij}^{\mu}(S) dS_{\mu} \right]$$

path ordered product

cycle in space of shapes  $S$

Infinitesimal rotation from infinitesimal  $\mu$  deformation

This is geometry. For  $A$ , need dynamics, i.e. angular momentum zero:

$$A_{ij}^{\mu} \frac{dS_{\mu}}{dt} = - \varepsilon_{ijk} (I - 1 \text{Tr} I)_{kl}^{-1} L_l$$

traceless inertia tensor of body in shape  $S$  in standard orientation

apparent angular momentum of deforming body when reduced to standard orientation

$A_{ij}^{\mu}$  is a vector in shape space (with axes  $\mu$ ).  
It is the <sup>(matrix-valued)</sup> potential of the gauge field

$$F^{\mu\nu} = \frac{\partial A^{\mu}}{\partial S_{\nu}} - \frac{\partial A^{\nu}}{\partial S_{\mu}} + [A^{\mu}, A^{\nu}]$$