Relativistic Theory of Rotating Disks

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Abstract—**The relationships between e, b, h, d in a rotating body are derived in the laboratory frame. They are utilized, together with suitable boundary conditions, to calculate the fields that arise when conducting or nonconducting bodies of revolution rotate in externally applied dc electric and magnetic fields.**

I. Introduction

Solution of electromagnetic problems involving rotating bodies is of fundamental importance for the electrical engineer concerned with rotating machinery. It is true that very efficient *ad hoc* formulas exist for designing rotating machines. A deeper understanding of the field problem, however, must rely on relativistic principles. Electrical engineers are becoming progressively aware of the importance of relativity as a central and essential part of electromagnetism. Progress in that direction has been hampered by the abstract character of the relativistic equations and by the lack of interest of experts in relativity for the technical aspects of their discipline. Particularly interesting for technical applications is the formalism of coordinate transformations. To investigate fields in rotating bodies, for example, one introduces a corotating system of coordinates and solves Maxwell's equations in the new coordinates [1]–[3]. The mathematical solution of these equations is quite complex, and it is our purpose to show how it can be avoided when the rotating body possesses symmetry of revolution. For such a case, it is possible, by means of fairly simple calculations, to obtain a solution in the (inertial) laboratory system. We have applied this method to the determination of the fields surrounding rotating disks immersed in simple electric or magnetic fields. We have considered, in particular, the metallic Faraday disks utilized in modern homopolar generators and the dielectric disks involved in the Wilson and Röntgen–Eichenwald experiments. These structures are of considerable practical interest, and their analysis by relativistic methods should fulfill a real didactic need.

II. Maxwell's Equations for a Body in Arbitrary Motion

A. *Field Transformation Formulas for Inertial Frames*

Our discussion is based on the field transformation formulas of special relativity. These formulas connect the electromagnetic fields in two inertial frames in relative (translational) motion [1], [4]. The inertial frames in our problem are the laboratory system S and the rest frame S' of a body undergoing a translatory motion with respect to S. In S' the electromagnetic fields satisfy Maxwell's equations:

$$\mathrm{curl}'\, \boldsymbol{e}' = -\frac{\partial \boldsymbol{b}'}{\partial t'}$$

$$\mathrm{curl}'\, \boldsymbol{h}' = \boldsymbol{j}' + \frac{\partial \boldsymbol{d}'}{\partial t'}$$

$$\mathrm{div}'\, \boldsymbol{d}' = \rho'$$

$$\mathrm{div}'\, \boldsymbol{b}' = 0. \tag{1}$$

Identical equations are valid in S, but the primes must now be dropped. In S' the constitutive relationships are those of a body at rest. We shall assume them to be of the form

$$\boldsymbol{d}' = \epsilon \boldsymbol{e}'$$
$$\boldsymbol{b}' = \mu \boldsymbol{h}'$$
$$\boldsymbol{j}' = \sigma \boldsymbol{e}'. \tag{2}$$

The constitutive relationships in S lose the simplicity that was characteristic of the rest frame. They can be derived by replacing $\boldsymbol{e}'$, $\boldsymbol{h}'$, $\boldsymbol{b}'$, $\boldsymbol{d}'$ in (2) by their expression in terms of $\boldsymbol{e}$, $\boldsymbol{h}$, $\boldsymbol{b}$, $\boldsymbol{d}$ at the corresponding point of S. Notice that corresponding points in S and S' are connected by the Lorentz transformation:

$$x = x'$$
$$y = y'$$
$$z = \frac{z' + vt'}{\sqrt{1-\beta^2}}$$
$$t = \frac{t' + \dfrac{vz'}{c^2}}{\sqrt{1-\beta^2}} \tag{3}$$

where $\beta = v/c$. The transformation formulas for the fields are classical [1]; they will not be reproduced here. From them we obtain the constitutive equations [4], [5]:

$$\boldsymbol{d} + \frac{\boldsymbol{v}\times\boldsymbol{h}}{c^2} = \epsilon(\boldsymbol{e} + \boldsymbol{v}\times\boldsymbol{b})$$

$$\boldsymbol{b} - \frac{\boldsymbol{v}\times\boldsymbol{e}}{c^2} = \mu(\boldsymbol{h} - \boldsymbol{v}\times\boldsymbol{d})$$

$$\boldsymbol{j} = \rho\boldsymbol{v} + \sigma\sqrt{1-\beta^2}\,\boldsymbol{e}_{\parallel} + \frac{\sigma}{\sqrt{1-\beta^2}}(\boldsymbol{e}_{\perp} + \boldsymbol{v}\times\boldsymbol{b}). \tag{4}$$

The symbol $\boldsymbol{e}_{\parallel}$ stands for the component parallel to the motion (i.e., $\boldsymbol{e}_z$ in Fig. 1), and $\boldsymbol{e}_{\perp}$ for the component perpendicular to z (i.e., $e_x\boldsymbol{u}_x + e_y\boldsymbol{u}_y$ in Fig. 1). It is useful, for our purposes, to express $\boldsymbol{d}$ and $\boldsymbol{b}$ in terms of $\boldsymbol{e}$ and $\boldsymbol{h}$. Elimination from (4) yields

$$\boldsymbol{d} - n^2\beta^2\boldsymbol{d}_{\perp} = \epsilon(\boldsymbol{e} - \beta^2\boldsymbol{e}_{\perp}) + \frac{n^2-1}{c^2}\boldsymbol{v}\times\boldsymbol{h}$$

$$\boldsymbol{b} - n^2\beta^2\boldsymbol{b}_{\perp} = \mu(\boldsymbol{h} - \beta^2\boldsymbol{h}_{\perp}) - \frac{n^2-1}{c^2}(\boldsymbol{v}\times\boldsymbol{e}) \tag{5}$$

where n is the index of refraction $(\epsilon_r\mu_r)^{1/2}$.

The transformation formulas for currents and charge

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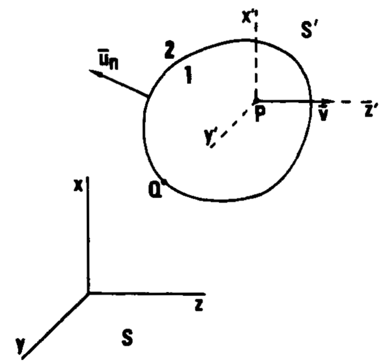

Fig. 1. A body and its rest frame S', in translational motion with respect to the inertial laboratory system S.

densities are also relevant to our discussion:

$$j_\parallel = \frac{j_\parallel' + \rho' v}{\sqrt{1-\beta^2}}$$
$$j_\perp = j_\perp'$$
$$\rho = \frac{\rho' + \dfrac{v j_\parallel'}{c^2}}{\sqrt{1-\beta^2}} . \tag{6}$$

B. Boundary Conditions

The boundary conditions in the rest frame are the classical relationships derived in any elementary physics text. The tangential component of $\boldsymbol{e}'$, for example, is continuous, and so is the normal component of $\boldsymbol{b}'$. Expressed in the laboratory frame, these conditions become [1], [4], [6]

$$\begin{aligned} \boldsymbol{u}_n \times (\boldsymbol{e}_2 - \boldsymbol{e}_1) - (\boldsymbol{u}_n \cdot \boldsymbol{v})(\boldsymbol{b}_2 - \boldsymbol{b}_1) &= 0 \\ \boldsymbol{u}_n \cdot (\boldsymbol{d}_2 - \boldsymbol{d}_1) &= \rho_s \\ \boldsymbol{u}_n \times (\boldsymbol{h}_2 - \boldsymbol{h}_1) + (\boldsymbol{u}_n \cdot \boldsymbol{v})(\boldsymbol{d}_2 - \boldsymbol{d}_1) &= \boldsymbol{j}_s \\ \boldsymbol{u}_n \cdot (\boldsymbol{b}_2 - \boldsymbol{b}_1) &= 0. \end{aligned} \tag{7}$$

Here ρ_s and $\boldsymbol{j}_s$ denote the surface-charge and current densities. We notice that the boundary conditions in S become the "rest" conditions at those points where the velocity lies in the tangent plane (i.e., where $\boldsymbol{u}_n \cdot \boldsymbol{v} = 0$).

C. Constitutive Equations in Accelerated Systems

Assume that a body with given ϵ, μ, δ undergoes an arbitrary motion, and that v_P is the instantaneous velocity in P. The formulation of the constitutive equations in the moving coordinates is a subject of current interest in the recent literature. The various approaches yield results which agree to first order in the velocities, but differ in the higher order terms, which are unfortunately very difficult to measure accurately. The general approach in this paper has been suggested by Professor Møller (the author of [2]) and is also to be found in a number of specialized articles [7]–[9]. It consists in stating that the constitutive equations of a body at rest, i.e., (2), hold in the *instantaneous rest frame* [2] of a point P, provided changes due to acceleration strain are neglected. The instantaneous rest frame S_P of P, which is the inertial frame having a velocity v_P with respect to the laboratory system S, varies from point to point and from instant to instant. This variation is known to influence the form of Maxwell's equations in the accelerated system, but this fact is of no importance to us, as our calculations are performed in the laboratory frame. We shall consequently state that the constitutive equations (2) hold in the instantaneous rest frame of P, and that the corresponding equations (4) in S are still valid, provided $\boldsymbol{v}$ is interpreted as the instantaneous velocity in P. Similarly, boundary conditions (7) hold at a boundary point Q, provided $\boldsymbol{v}$ is taken to be the instantaneous velocity in Q. These principles have recently been applied by Mo to the scattering of a plane wave by a nonconducting rotating sphere [9]. In this paper, we shall consider their application to situations that are of more direct interest to power engineers. These situations have been selected on the grounds of their mathematical simplicity, but their analysis can serve as a guide to the solution of more complex problems.

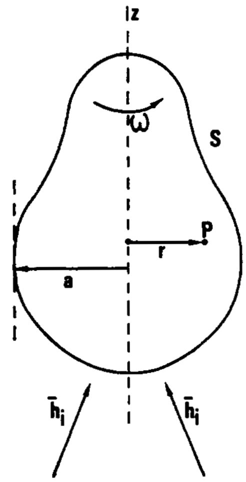

Fig. 2. A conducting body of revolution rotating with angular velocity ω in an externally applied electric field.

III. A Conductor Rotating in a DC Magnetic Field: Symmetry of Revolution

A. Constitutive Equations in the Rotating Body

In this paragraph we assume that all fields depend on r and z but are independent of the azimuth φ and time t (Fig. 2). The electric field satisfies curl $\boldsymbol{e} = 0$ throughout space. Its azimuthal component e_φ is therefore zero because the circulation of $\boldsymbol{e}$ around a circle centered on the z axis vanishes. It follows that $\boldsymbol{e}$ is a meridian vector and that it can be written as

$$\boldsymbol{e} = -\operatorname{grad}\phi \tag{8}$$

where ϕ is a function of r and z. To determine ϕ (and the other electromagnetic quantities) we shall make use of the equations

$$\begin{aligned} \operatorname{curl}\boldsymbol{h} = \boldsymbol{j} &= \frac{\sigma}{\sqrt{1-\beta^2}}(\boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b}) + \boldsymbol{u}_\varphi v \operatorname{div}\boldsymbol{d} \\ \operatorname{div}\boldsymbol{b} &= 0 \end{aligned} \tag{9}$$

where $v = \omega r$ and where the expression for the current density is taken from (4). Outside the body, $\boldsymbol{h}$ is irrotational, and $\boldsymbol{e}$ and $\boldsymbol{h}$ are regular at infinity. Notice that $\beta = \omega r/c$ is a function of r. Direct solution of (9) is difficult because of the complicated nature of the constitutive equations (5). An iterative solution, on the other hand, is quite manageable. In all practical applications the maximum peripheric velocity ωa is completely negligible with respect to the velocity of light. The parameter $\beta_a = \omega a/c$ is therefore very small, and we shall utilize it to expand the fields in the following manner:

$$\begin{aligned} h &= h_0 + \beta_a h_1 + \beta_a^2 h_2 + \cdots \\ e &= \beta_a e_1 + \beta_a^2 e_2 + \cdots . \end{aligned} \tag{10}$$

The field h_0 is the static value which exists in the absence of rotation. It is a function of the geometry and magnetic permeability of the body immersed in the applied field h_i. We shall assume, for simplicity, that h_i and h_0 lie in the meridian plane. The electric field vanishes in the nonrotating body, and therefore its expansion starts with a first-order term. Inserting the expansions for e and h in the constitutive equations (5) yields the power expansions for d and b:

$$\begin{aligned} d &= \beta_a \left[\epsilon e_1 + (n^2 - 1) \frac{r}{ac} u_\varphi \times h_0 \right] \\ &\quad + \beta_a^2 \left[\epsilon e_2 + (n^2 - 1) \frac{r}{ac} u_\varphi \times h_1 \right] \\ b &= \mu h_0 + \beta_a \mu h_1 + \beta_a^2 \left[\mu h_2 + \mu (n^2 - 1) \frac{r^2}{a^2} h_0 \right. \\ &\quad \left. - (n^2 - 1) \frac{r}{ac} u_\varphi \times e_1 \right]. \end{aligned} \tag{11}$$

B. Electric and Magnetic Fields in the Rotating Body

Having obtained the expansions for d and b, we introduce these expressions into Maxwell's equations (9) and equate terms of equal order in β_a on both sides of the equations. This generates a set of iterative equations. The same procedure can be applied to the boundary conditions, which are of the "rest" type because the velocity at a boundary point is tangential to the body. It should be noticed that the volume and surface convection currents ρv and $\rho_s v$ are of order β_a^2, as d is of order β_a. It follows that the tangential component of h_1 across S is continuous, as there is no surface current of order one there. These various remarks lead to the following equations for the first-order magnetic field h_1:

$$\begin{aligned} &\text{curl } h_1 = j_1 = \sigma \left(e_1 + \frac{r}{a} \mu c u_\varphi \times h_0 \right) \text{ in the conductor} \\ &\text{curl } h_1 = 0 \text{ outside the conductor} \\ &\text{div } h_1 = 0 \\ &h_1 \text{ regular at infinity} \\ &(h_1)_{\tan} \text{ continuous across } S. \end{aligned} \tag{12}$$

These equations imply that the first-order current density j_1 vanishes. We notice, indeed, that div j_1 and j_{1n} are zero, as in every time-independent current flow. To prove that j_1 vanishes, it suffices to show that curl j_1 is zero. The proof is immediate if we take the curl of both members of the first equation in (12). Thus

$$\text{curl } j_1 = \sigma \left[\text{curl } e_1 + \frac{\mu c}{a} \text{curl } (r u_\varphi \times h_0) \right]. \tag{13}$$

The term curl e_1 vanishes because e is irrotational, the fields being time independent. The second term also vanishes because

$$\text{curl } (r u_\varphi \times h_0) = u_\varphi r \text{ div } h_0 = 0. \tag{14}$$

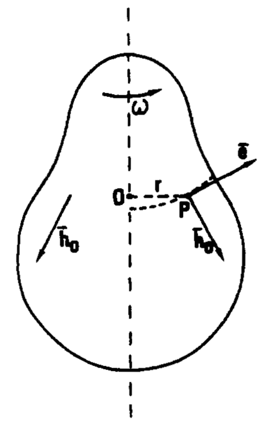

Fig. 3. Lines of force of the electric field in a conductor.

It follows that j_1 is zero in the body of revolution shown in Fig. 2.

The absence of volume currents j_1 has interesting consequences. First, h_1 and b_1 vanish everywhere, as can be seen from (11) and (12). The same equations imply that

$$e_1 = -\frac{\mu c}{a} r u_\varphi \times h_0 = +\frac{\mu c}{a} r(-h_{0z} u_r + h_{0r} u_z) \tag{15}$$

in the conductor. The leading term for the electric field is therefore

$$\begin{aligned} \beta_a e_1 = \frac{\omega a}{c} e_1 &= -v \times \mu h_0 = -v \times b_0 \\ &= -\omega r (b_{0z} u_r - b_{0r} u_z). \end{aligned} \tag{16}$$

Fig. 3 shows a typical line of force in the rotating body. As the electric field vanishes along the axis of revolution, the potential must have a constant value $\phi(0)$ there. The potential in an arbitrary point P is given by

$$\phi(P) - \phi(0) = \int_P^0 e \cdot dr = \omega \int_0^P r b_{0z} dr. \tag{17}$$

To determine e and ϕ outside the body (and in particular $\phi(0)$), it is necessary to solve an exterior potential problem. An example of application will be given in the next paragraph. We leave this problem aside and notice that we are now in a position to determine d_1, as e_1 and h_1 are known. From the knowledge of d_1, first-order volume and surface-charge densities can be determined. The explicit calculations are based on (11). They yield

$$\begin{aligned} d &= -\beta_a \frac{r}{ac} u_\varphi \times h_0 + \beta_a^2 \epsilon e_2 + \cdots \\ &= -\frac{\omega r}{c^2} u_\varphi \times h_0 + \beta_a^2 \epsilon e_2 + \cdots \end{aligned} \tag{18}$$

from which we deduce that

$$\rho = \text{div } d = -\frac{2\omega h_{0z}}{c^2} + \beta_a^2 \epsilon \text{ div } e_2.$$

The discontinuity in the normal components of d gives ρ_s. This value must be known to derive the discontinuity in the tangential component of h_2. We shall not dwell upon this point, but will try to evaluate $\boldsymbol{e}_2$. We start from

$$\operatorname{curl} \boldsymbol{h}_2 = \boldsymbol{j}_2 = \sigma \boldsymbol{e}_2 - 2\frac{r h_{0z}}{a^2}\boldsymbol{u}_\varphi$$

which is a direct consequence of (9). Taking the divergence of both members shows that div $\boldsymbol{e}_2 = 0$. This equation is valid throughout space. The same is true of curl $\boldsymbol{e}_2 = 0$. Furthermore, the tangential component of $\boldsymbol{e}_2$ is continuous across S. We conclude that $\boldsymbol{e}_2$ vanishes everywhere.

The results obtained until now can be summarized in the following six formulas:

$$\begin{aligned}
\boldsymbol{e} &= -\,\omega r(\boldsymbol{u}_\varphi \times \boldsymbol{b}_0) + \text{terms of order } \beta_a^3\\
\boldsymbol{h} &= \boldsymbol{h}_0 + \text{terms of order } \beta_a^2\\
\boldsymbol{d} &= -\frac{\omega r}{c^2}\boldsymbol{u}_\varphi \times \boldsymbol{h}_0 + \text{terms of order } \beta_a^3\\
\boldsymbol{b} &= \mu \boldsymbol{h}_0 + \mu\beta_a^2 \boldsymbol{h}_2 + \text{terms of order } \beta_a^3\\
\rho &= -\frac{2\omega h_0 z}{c^2} + \text{terms of order } \beta_a^3\\
\boldsymbol{j} &= -\,2\frac{\omega^2 r}{c^2} h_{0z}\boldsymbol{u}_\varphi + \text{terms of order } \beta_a^3.
\end{aligned} \tag{19}$$

IV. Application to Rotating Conducting Spheres and Spheroids: The Faraday Disk

A. Fields In and Around a Rotating Conducting Sphere

Consider a sphere in a uniform magnetic field parallel to the z axis (Fig. 4). The magnetic field in the sphere is proportional to $\boldsymbol{h}_i$ and is given by

$$\boldsymbol{h}_0 = \frac{3}{\mu_r + 2} h_i \boldsymbol{u}_z \tag{20}$$

where μ_r is the permeability of the sphere. Application of (19) gives fields and charges *inside* the sphere. To calculate the first-order electric field *outside* the sphere, we set $\boldsymbol{e}_1 = -\operatorname{grad} \phi_1$ and write, from (15) and (17),

$$\begin{aligned}
\nabla^2\phi_1 &= 0 \text{ everywhere}\\
\phi_1(A) &= \phi_1(0) + \frac{\mu c}{2} h_0 a \sin^2\theta \text{ at the surface}\\
\iint_S \frac{\partial \phi_1}{\partial R} dS &= 2\pi a^2 \int_0^\pi \left.\frac{\partial \phi_1}{\partial R}\right|_A \sin\theta d\theta = 0\\
\phi_1 &\text{ regular at infinity.}
\end{aligned} \tag{21}$$

The third condition expresses the charge neutrality of the sphere. It is this condition which makes the solution mathematically unique and allows determination of $\phi_1(0)$. In problems where the only quantity of interest is the potential difference between a surface point A and the axis, the answer is immediately afforded by the second equation of (21), viz.,

$$\begin{aligned}
\phi(A) - \phi(0) &= \beta_a[\phi_1(A) - \phi_1(0)]\\
&= \frac{\omega b_0}{2} a^2 \sin^2\theta = \frac{\omega b_0}{2} r_A^2.
\end{aligned} \tag{22}$$

This formula often appears in the literature, where it is typically derived by nonrelativistic methods involving a

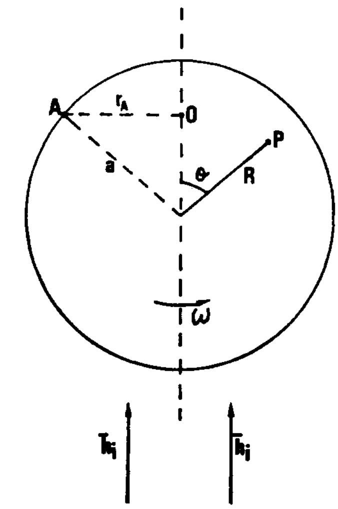

Fig. 4. Rotating sphere in a uniform magnetic field parallel to the rotation axis.

power-series expansion in a time-rate parameter [10]. If more information than (22) is needed, solution of (21) must be effected. This can be done by means of a Legendre polynomial expansion:

$$\phi_1 = \sum P_n(\cos\theta)\frac{B_n}{R^{n+1}}\,.$$

It is only necessary to keep the term in

$$P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1) = 1 - \tfrac{3}{2}\sin^2\theta$$

to satisfy all conditions appearing in (21). A few trivial calculations yield

$$\phi_1 = -\frac{a^4 c b_0}{3R^3}(1 - \tfrac{3}{2}\sin^2\theta) \tag{23}$$

from which we deduce that

$$\phi_1(0) = -\frac{acb_0}{3}$$

$$d_R = -\,\epsilon_0\beta_a\frac{\partial\phi_1}{\partial R} = -\,\omega a\epsilon_0 b_0(1 - \tfrac{3}{2}\sin^2\theta).$$

This value of d_R is valid just outside the body. The value just inside is, from (18),

$$d_R = \boldsymbol{d}\cdot\boldsymbol{u}_R = -\frac{\omega r}{c^2} h_0(\boldsymbol{u}_r\cdot\boldsymbol{u}_R) = -\frac{\omega}{c^2} a h_0 \sin^2\theta. \tag{24}$$

The surface-charge density is equal to the discontinuity of d_R and is, therefore,

$$\rho_s = \omega\epsilon_0 a b_0\left(\frac{3}{2}\sin^2\theta + \frac{1}{\mu_r}\sin^2\theta - 1\right). \tag{25}$$

This surface charge generates a surface current $\rho_s \omega r \boldsymbol{u}_\varphi$, which in turn produces a discontinuity in the tangential component of $\boldsymbol{h}_2$ on S, namely,

$$\begin{aligned}
\boldsymbol{h}_{2\,\tan}{}^{\text{ext}} - \boldsymbol{h}_{2\,\tan}{}^{\text{int}} = \boldsymbol{j}_{s2} \times \boldsymbol{u}_r = \sin\theta h_0(\tfrac{3}{2}\mu\sin^2\theta\\
+ \sin^2\theta - \mu_r)\boldsymbol{u}_\theta.
\end{aligned} \tag{26}$$

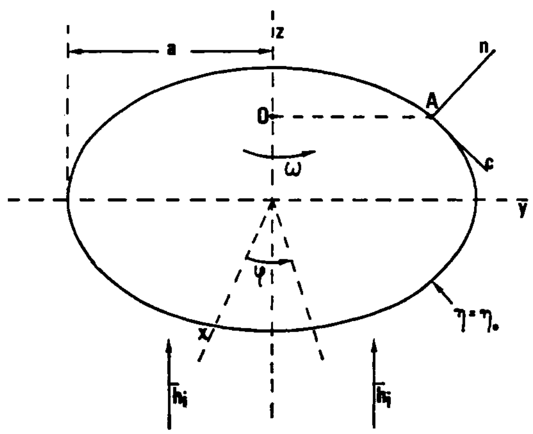

Fig. 5. Oblate spheroid rotating about its short axis; applied magnetic field parallel to the rotation axis.

This boundary condition, completed by the relationships

$$\begin{aligned}
\operatorname{curl} \boldsymbol{h}_2 &= -2\frac{rh_0}{c^2}\boldsymbol{u}_\varphi \text{ in the conductor}\\
\operatorname{curl} \boldsymbol{h}_2 &= 0 \text{ outside the conductor}\\
\operatorname{div} \boldsymbol{h}_2 &= 0\\
\boldsymbol{h}_2 &\text{ regular at infinity}
\end{aligned} \tag{27}$$

allows determination of $\boldsymbol{h}_2$ everywhere in space. The solution is trivial and will not be given in any detail. It is based on the knowledge of a particular solution $h_0(r^2/a^2)\boldsymbol{u}_z$ of the first equation.

B. Fields In and Around a Rotating Conducting Spheroid

Consider next an oblate spheroid rotating about the z axis and immersed in a uniform magnetic field parallel to this axis. A solution by separation of variables can be effected [11], [12] by utilizing spheroidal coordinates (Fig. 5):

$$\begin{aligned}
x &= f\cosh\eta\sin\theta\cos\varphi\\
y &= f\cosh\eta\sin\theta\sin\varphi\\
z &= f\sinh\eta\cos\theta.
\end{aligned} \tag{28}$$

The surface of the spheroid is defined by $\eta=\eta_0$. The maximum radius is $a=f\cosh\eta_0$. The magnetic field in the conductor is uniform and parallel to the z axis, and is given by

$$\boldsymbol{h}_0 = \frac{\boldsymbol{h}_i}{(\mu_r-1)g_\parallel+1} \tag{29}$$

where [12]

$$g_\parallel = \cosh^2\eta_0(1-\sinh\eta_0\cot^{-1}\sinh\eta_0). \tag{30}$$

The solution for ϕ_1 proceeds as for the sphere. The potential law on the spheroid is, from (17),

$$\phi_1(A) = \phi_1(0) + \frac{cb_0}{2}f\cosh\eta_0\sin^2\theta. \tag{31}$$

To match the outside potential to this value, we utilize the classical expansion

$$\phi = \sum_{m,n} A_n{}^m P_n{}^m(\cos\theta)Q_n{}^m(j\sinh\eta)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\varphi$$

where $P_n{}^m$ and $Q_n{}^m$ are the associated Legendre polynomials of the first and second kind. It turns out that all requirements are satisfied by a single term

$$\phi_1 = AP_2(\cos\theta)Q_2(j\sinh\eta)$$

where

$$Q_2(jz) = \frac{j}{2}\left[(3z^2+1)\cot^{-1}z - 3z\right].$$

A few trivial calculations yield, for the potential *outside* the spheroid,

$$\phi_1{}^{\text{ext}} = -\frac{b_0c}{3}f\cosh\eta_0\left(1-\frac{3}{2}\sin^2\theta\right) \cdot \frac{(3\sinh^2\eta+1)\cot^{-1}\sinh\eta - 3\sinh\eta}{(3\sinh^2\eta_0+1)\cot^{-1}\sinh\eta_0 - 3\sinh\eta_0}. \tag{32}$$

Let us check that this value of the potential satisfies the charge neutrality conditions. For very large values of η, the spheroidal coordinates become spherical coordinates, and $\phi_1{}^{\text{ext}}$ becomes proportional to $P_2(\cos\theta)(1/R^3)$. The flux of the electric field through a large sphere of radius R is therefore zero, because $\iint P_2(\cos\theta)\,d\Omega = \iint P_2(\cos\theta)\sin\theta\,d\theta\,d\varphi = 0$. As the flux of $\boldsymbol{e}$ is conservative, we conclude that the charge neutrality condition of the spheroid is satisfied.

The potential *inside* the conductor is

$$\phi_1{}^{\text{int}} = \frac{b_0cf}{2\cosh\eta_0}\cosh^2\eta\sin^2\theta - \underbrace{\frac{b_0c}{3}f\cosh\eta_0}_{\phi_1(0)}. \tag{33}$$

Knowledge of the potentials outside and inside the surface allows determination of the surface-charge density through the formula

$$\rho_s = \left[-\epsilon_0\frac{\partial\phi_1{}^{\text{ext}}}{\partial n} + \frac{\epsilon_0}{\mu_r}\frac{\partial\phi_1{}^{\text{int}}}{\partial n}\right]\frac{\omega}{c}f\cosh\eta_0. \tag{34}$$

This equation is based on the relationship

$$\boldsymbol{d}_1 = \frac{\epsilon_0}{\mu_r}\boldsymbol{e}_1 \tag{35}$$

which holds inside the conductor, as can easily be deduced from (19). If it is desired to calculate ρ_s, the derivatives with respect to n should be evaluated with the help of the formula for the increment of length along the normal:

$$dn = f\sqrt{\sinh^2\eta+\cos^2\theta}\,d\eta. \tag{36}$$

C. Fields Around a Faraday Disk

A very flat disk is obtained by letting η_0 approach zero (Fig. 6). The resulting structure is a Faraday disk [13]. The potential outside the conductor is

$$\phi_1{}^{\text{ext}} = -\frac{2}{3\pi}b_0ac\left(1-\frac{3}{2}\sin^2\theta\right) \cdot \left[(3\sinh^2\eta+1)\cot^{-1}\sinh\eta - 3\sinh\eta\right]. \tag{37}$$

In the vicinity of the disk (i.e., for small values of η), the external potential takes the form

$$\phi_1{}^{\text{ext}} = -\frac{b_0ac}{3}\left(1-\frac{8\eta}{\pi}\right)\left(1-\frac{3}{2}\sin^2\theta\right). \tag{38}$$

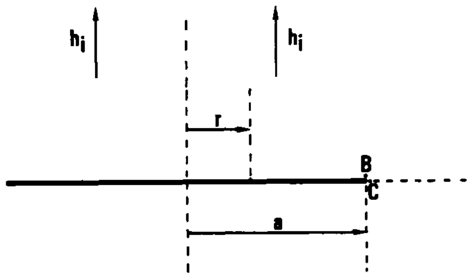

Fig. 6. Faraday disk.

From (36),

$$\frac{\partial \phi}{\partial n} = \frac{1}{f\,|\cos\theta|}\,\frac{\partial \phi}{\partial n}\,.$$

It follows that the charge density at $r = a \sin\theta$ is given by

$$\rho_s = -\,2\epsilon_0 \frac{\omega_a}{c}\,\frac{\partial \phi_1^{\text{ext}}}{\partial n} = \epsilon_0 \frac{\omega b_0}{a}\,\frac{8}{\pi}\left(r^2 - \frac{2}{3}a^2\right) \cdot \left(1 - \frac{r^2}{a^2}\right)^{-1}, \quad \text{for } r \leq f. \tag{39}$$

The density becomes infinite near the edge of the disk (point B in Fig. 6). A discontinuity of the same kind occurs in the equatorial plane for points close to the disk (point C in Fig. 6). In that plane, which is defined by $\theta = \pi/2$,

$$\phi_1^{\text{ext}} = \frac{b_0 a c}{3\pi}\left[(3\sinh^2\eta + 1)\cot^{-1}\sinh\eta - 3\sinh\eta\right]$$

$$r = f\sqrt{1 + \sinh^2\eta}.$$

As η approaches zero,

$$\rho_s = \lim_{\eta\to 0}\left(-\epsilon_0 \frac{\omega a}{c}\,\frac{\partial \phi_1^{\text{ext}}}{\partial n}\right) = \epsilon_0 \frac{\omega b_0}{a}\,\frac{4}{3\pi}\left(\frac{r^2}{a^2} - 1\right)^{-1}, \quad \text{for } r \geq f. \tag{40}$$

This equation confirms that the charge density is infinite at the edge.

V. Rotating Dielectric in a Magnetic Field: Symmetry of Revolution; The Wilson Experiment

We assume that the dielectric is nonconducting and does not carry any fixed real charge. In the local rest frame both j' and ρ' are therefore zero. It follows, from the transformation equations (6), that j and ρ also vanish in the laboratory frame. Because of the assumed symmetry of revolution, phenomena are time independent, and Maxwell's equations inside the dielectric take the form

$$\begin{aligned} \operatorname{curl} \boldsymbol{e} &= 0 \\ \operatorname{curl} \boldsymbol{h} &= 0 \\ \operatorname{div} \boldsymbol{d} &= 0 \\ \operatorname{div} \boldsymbol{b} &= 0. \end{aligned} \tag{41}$$

The first two equations show that $\boldsymbol{e}$ and $\boldsymbol{h}$ can be derived from scalar potentials. Thus

$$\begin{aligned} \boldsymbol{e} &= -\operatorname{grad}\phi \\ \boldsymbol{h} &= -\operatorname{grad}\psi. \end{aligned} \tag{42}$$

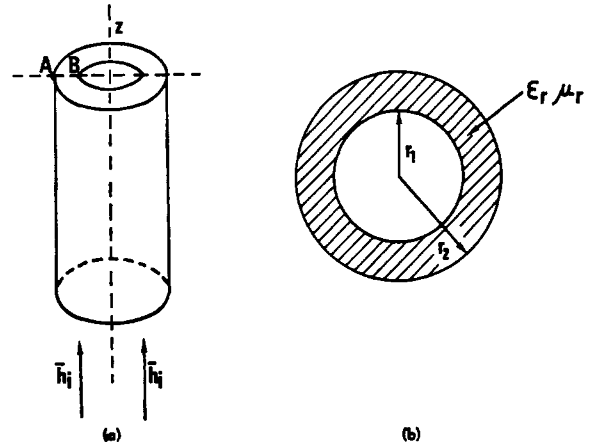

Fig. 7. Wilson's experiment. (a) Hollow dielectric cylinder rotating in a magnetic field parallel to the axis. (b) Cross-sectional view.

Differential equations for ϕ and ψ are obtained by making use of the divergence equations (41) and the constitutive equations (5). Methods similar to those described in Section III can be applied here. We shall consider a particular example—the experiment of Wilson [1], [14]—in which a hollow dielectric cylinder rotates in a magnetic field $\boldsymbol{h}_i$ parallel to the rotation axis (Fig. 7). As a result, a difference of potential appears between points A and B. We shall calculate this difference of potential by assuming the cylinder to be infinite. Phenomena are now independent of φ and z, and depend only on r. Fields $\boldsymbol{b}$ and $\boldsymbol{h}$ are parallel to the z axis. From curl $\boldsymbol{h} = 0$ we deduce that $\partial h_z/\partial r = 0$, i.e., that h_z is constant throughout space. We therefore set $\boldsymbol{h} = h_0 \boldsymbol{u}_z$. The electric field is radial, and (5) shows that the same is true for the induction $\boldsymbol{d}$. In the absence of any real charge, $\boldsymbol{d}$ must be zero everywhere, and we deduce from (5) that

$$\boldsymbol{e} = -\frac{n^2 - 1}{1 - \beta^2}\,\frac{\boldsymbol{v} \times \boldsymbol{h}}{\epsilon c^2}$$

i.e.,

$$e_r = -\left(\mu_r - \frac{1}{\epsilon_r}\right)\mu_0 h_0 \frac{\omega r}{1 - \dfrac{\omega^2 r^2}{c^2}}\,. \tag{43}$$

The difference of potential between A and B is therefore

$$\phi_A - \phi_B = \left(\mu_r - \frac{1}{\epsilon_r}\right)\mu_0 h_0 \frac{c^2}{2\omega}\log_e \frac{1 - \dfrac{\omega^2}{c^2} r_1^2}{1 - \dfrac{\omega^2}{c^2} r_2^2}\,. \tag{44}$$

For small velocities,

$$\begin{aligned} \phi_A - \phi_B &= \left(\mu_r - \frac{1}{\epsilon_r}\right)\mu_0 h_0 \omega (r_2^2 - r_1^2) \\ &= \frac{\omega}{2\pi}\left(\mu_r - \frac{1}{\epsilon_r}\right)\Phi \end{aligned} \tag{45}$$

where Φ is the flux of $\mu_0 h_0$ through the ring. We illustrate (45) by a short numerical example. Consider the following values:

$$\mu_r = 3, \epsilon_r = 6, r_1 = 2\text{ cm}, r_2 = 4\text{ cm}$$
$$\omega/2\pi = 100\text{ r/s}, \mu_0 h_0 = 0.125\text{ Wb}\cdot\text{m}^{-2}.$$

The resulting difference of potential is 0.13 V.

To complete the field calculation we give the formula for $\boldsymbol{b}$ inside the dielectric. By combining (5) and (43) we obtain

$$\boldsymbol{b} = \mu_0\mu_r h_0 \frac{1 - \dfrac{\omega^2 r^2}{\epsilon_r\mu_r c^2}}{1 - \dfrac{\omega^2 r^2}{c^2}} \boldsymbol{u}_z. \tag{46}$$

It is seen that the rotation results in an apparent increase of the magnetic permeability.

The reader will notice that the values derived for the fields are valid for arbitrary angular velocities ω, i.e., that an expansion in the small parameter β_a is not needed for the structure under study. This interesting result obtains because the cylinder was assumed to be infinite, an assumption which reduced the number of independent variables of the problem. For dielectric cylinders and bodies of finite length, the solution is best effected by utilizing an expansion in the velocity parameter β_a, as in Section III. In Section VI we apply this method to calculate the fields in a dielectric body rotating in a dc electric field.

VI. Dielectric Body Rotating in a DC Electric Field: Symmetry of Revolution; The Röntgen–Eichenwald Experiment

A. General Remarks

In the Röntgen–Eichenwald experiment, a dielectric disk rotates in the electric field produced by the armatures of a capacitor (Fig. 8). This motion generates an (induced) magnetic field [14]. To analyze the experiment, we first consider a general body of revolution rotating in an external meridian electric field $\boldsymbol{e}_i$ (Fig. 9). Let $\boldsymbol{e}_0$ be the value of the electric field in the dielectric when the latter is stationary. We introduce the power expansions

$$\begin{aligned} \boldsymbol{e} &= \boldsymbol{e}_0 + \beta_a \boldsymbol{e}_1 + \beta_a^2 \boldsymbol{e}_2 + \cdots \\ \boldsymbol{h} &= \beta_a \boldsymbol{h}_1 + \beta_a^2 \boldsymbol{h}_2 + \cdots. \end{aligned} \tag{47}$$

Insertion into (5) gives, inside the dielectric,

$$\begin{aligned} \boldsymbol{d} &= \epsilon \boldsymbol{e}_0 + \beta_a \epsilon \boldsymbol{e}_1 + \beta_a^2 \left[\epsilon \boldsymbol{e}_2 + \epsilon(n^2 - 1)\frac{r^2}{a^2}\boldsymbol{e}_0 + \frac{n^2 - 1}{c}\frac{r}{a}\boldsymbol{u}_\phi \times \boldsymbol{h}_1 \right] + \cdots \\ \boldsymbol{b} &= \beta_a \left[\mu \boldsymbol{h}_1 - \frac{n^2 - 1}{c}\frac{r}{a}\boldsymbol{u}_\phi \times \boldsymbol{e}_0 \right] \\ &\quad + \beta_a^2 \left[\mu \boldsymbol{h}_2 - \frac{n^2 - 1}{c^2}\frac{r}{a}\boldsymbol{u}_\phi \times \boldsymbol{e}_1 \right] + \cdots. \end{aligned} \tag{48}$$

A single look at these equations shows, in conjunction with (41), that $\boldsymbol{e}_1$ and $\boldsymbol{h}_2$ must vanish as their divergence and curl are zero. The magnetic field $\boldsymbol{h}_1 = -\text{grad}\, \psi_1$ does not vanish. Its potential ψ_1 is determined by means of the equation div $\boldsymbol{b} = 0$, which gives

$$\begin{aligned} \nabla^2 \psi_1^{\text{int}} &= -\frac{2(n^2 - 1)}{\mu a c} e_{0z} \\ \nabla^2 \psi_1^{\text{ext}} &= 0. \end{aligned} \tag{49}$$

The inside and outside potentials are connected by the boundary conditions on (C):

$$\begin{aligned} \psi_1^{\text{ext}} &= \psi_1^{\text{int}} \\ -\mu_0 \frac{\partial \psi_1^{\text{ext}}}{\partial n} &= -\mu\frac{\partial \psi_1^{\text{int}}}{\partial n} + \frac{n^2 - 1}{c}\frac{r}{a}(\boldsymbol{u}_c \cdot \boldsymbol{e}_0). \end{aligned} \tag{50}$$

These conditions are an expression of the continuity of the tangential component of $\boldsymbol{h}_1$ and the normal component of $\boldsymbol{b}_1$, respectively.

B. Rotating Dielectric Sphere

The application to a dielectric sphere plunged into a preexisting field $e_i \boldsymbol{u}_z$ proceeds much as in Section IV. The field inside the sphere is

$$\boldsymbol{e}_0 = \frac{3}{\epsilon_r + 2} e_i \boldsymbol{u}_z$$

where ϵ_r is the dielectric constant of the sphere. The calcula-

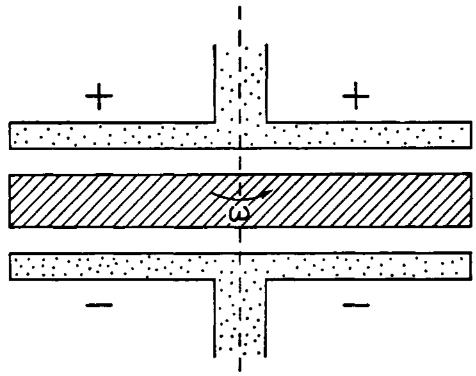

Fig. 8. Röntgen–Eichenwald experiment. Dielectric disk rotating in an external electric field.

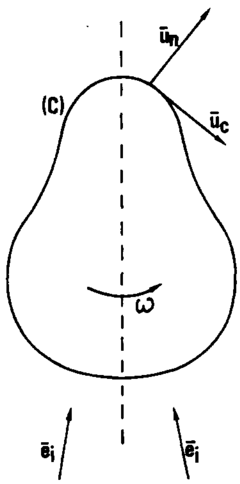

Fig. 9. A dielectric body of revolution rotating in an externally applied electric field.

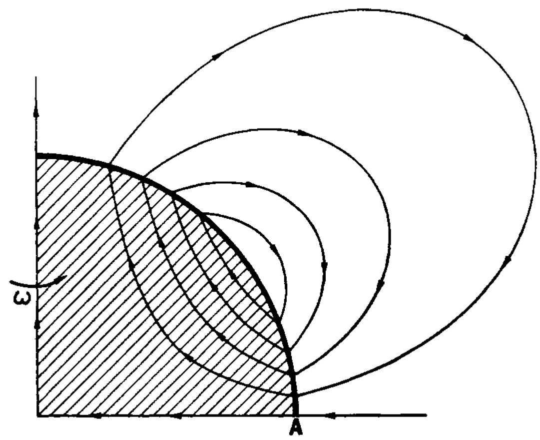

Fig. 10. Lines of force of the induction b produced by a dielectric sphere rotating in an electric field parallel to the rotation axis.

tion rests on the knowledge of a particular solution for the differential equation (49) satisfied by ψ_1^{int}. In spherical coordinates, it is

$$-\frac{(n^2-1)}{2\mu ac}R^2\sin^2\theta e_0.$$

We shall not dwell upon the details of the calculation, but will quote the final results:

$$\begin{aligned}
h^{\text{int}} &= \frac{\omega e_0}{c^2}R\frac{2(n^2-1)}{\mu(3+2\mu_r)} \\
&\quad\cdot[(1+\mu_r\sin^2\theta)\boldsymbol{u}_R+\sin\theta\cos\theta\boldsymbol{u}_\theta] \\
h^{\text{ext}} &= \frac{\omega e_0}{c^2}\frac{a^5}{R^4}\frac{2(n^2-1)}{\mu_0(3+2\mu_r)} \\
&\quad\cdot[(1-\tfrac{3}{2}\sin^2\theta)\boldsymbol{u}_R+\sin\theta\cos\theta\boldsymbol{u}_\theta] \\
b^{\text{int}} &= \frac{\omega e_0}{c^2}R\frac{2(n^2-1)}{3+2\mu_r} \\
&\quad\cdot[(1-\tfrac{3}{2}\sin^2\theta)\boldsymbol{u}_R-\tfrac{3}{2}\sin\theta\cos\theta\boldsymbol{u}_\theta] \\
b^{\text{ext}} &= \frac{\omega e_0}{c^2}\frac{a^5}{R^4}\frac{2(n^2-1)}{3+2\mu_r} \\
&\quad\cdot[(1-\tfrac{3}{2}\sin^2\theta)\boldsymbol{u}_R+\sin\theta\cos\theta\boldsymbol{u}_\theta].
\end{aligned}\tag{51}$$

Fig. 10 displays some of the lines of force of b for a quadrant of the sphere. These lines generate surfaces of revolution. The surfaces have been chosen, on the figure, to include equal amounts of magnetic flux. Their determination is based on the formula for the magnetic flux flowing between the rotation axis and a circle of polar angle θ on the sphere. This flux is

$$\Phi = \int\int_0^\theta b_R\sin\theta'\,d\theta'\,d\varphi' \doteq \cos\theta-\cos^3\theta.$$

The induction in A, just outside the equator, is given by

$$b = -\frac{\epsilon_r\mu_r-1}{2(3+2\mu_r)}\frac{\omega V}{c^2}\boldsymbol{u}_R$$

where V is the potential difference $2ae_0$ across the rotation axis. A sample calculation follows. At 100 r/s and with $\mu_r=1$, $\epsilon_r=3$, and $V=10$ kV, the induction at the equator is 1.4×10^{-11} Wb·m^{-2} $=1.4\times10^{-7}$ G, a very small value indeed.

C. Rotating Dielectric Spheroid

Similar calculations can be performed for an oblate spheroid immersed in a primary field $e_i\boldsymbol{u}_z$. The internal electric field is now

$$e_0 = \frac{e_i}{(\epsilon_r-1)g_\parallel+1}\boldsymbol{u}_z$$

where $g_\parallel$ has the value mentioned in (30). To apply the boundary conditions (50), it will be necessary to calculate $\partial\phi_0/\partial c$. This is done by utilizing the formula

$$dc = f\sqrt{\sinh^2\eta+\cos^2\theta}\,d\theta. \tag{52}$$

After quite a bit of algebra, one arrives at the following value for a typical field component, e.g., the normal component of b on the spheroid:

$$b_n = \frac{\omega e_0}{c^2}\left(1-\frac{3}{2}\sin^2\theta\right) \cdot\frac{2(n^2-1)f\sinh\eta_0\sinh^3\eta_0\alpha}{\sqrt{\sinh^2\eta_0+\cos^2\theta}[(3\sinh^2\eta_0+1)\alpha-6\mu_r\gamma\sinh\eta_0]}$$

where

$$\begin{aligned}
\alpha &= 6\sinh\eta_0\cot^{-1}\sinh\eta_0-3-\frac{3\sinh^2\eta_0+1}{\cosh^2\eta_0} \\
\gamma &= (3\sinh^2\eta_0+1)\cot^{-1}\sinh\eta_0-3\sinh\eta_0.
\end{aligned}\tag{53}$$

For a thin disk,

$$b_n = \frac{\omega e_0}{c^2}2(n^2-1)(1-\tfrac{3}{2}\sin^2\theta)\frac{ab}{\sqrt{b^2+a^2\cos^2\theta}}$$

where a and b are the semiaxes of the ellipse.

VII. Conclusions

The essence of this paper is contained in Section II-C, in which the constitutive equations and boundary conditions are stated for a rotating body. The remainder of the paper is merely an application of these simple rules to situations of physical interest, in which various disks rotate in various dc electric and magnetic fields. The mathematical solution has been pushed sufficiently far to arrive at useful formulas. The methods involved are of a classical nature—expansion in a (small) velocity parameter $\omega a/c$, and reduction of the solution of an iterative sequence of potential problems.

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The Viterbi Algorithm

G. DAVID FORNEY, JR.

Invited Paper

***Abstract*—The Viterbi algorithm (VA) is a recursive optimal solution to the problem of estimating the state sequence of a discrete-time finite-state Markov process observed in memoryless noise. Many problems in areas such as digital communications can be cast in this form. This paper gives a tutorial exposition of the algorithm and of how it is implemented and analyzed. Applications to date are reviewed. Increasing use of the algorithm in a widening variety of areas is foreseen.**

I. Introduction

THE VITERBI algorithm (VA) was proposed in 1967 [1] as a method of decoding convolutional codes. Since that time, it has been recognized as an attractive solution to a variety of digital estimation problems, somewhat as the Kalman filter has been adapted to a variety of analog estimation problems. Like the Kalman filter, the VA tracks the state of a stochastic process with a recursive method that is optimum in a certain sense, and that lends itself readily to implementation and analysis. However, the underlying process is assumed to be finite-state Markov rather than Gaussian, which leads to marked differences in structure.

This paper is intended to be principally a tutorial introduction to the VA, its structure, and its analysis. It also purports to review more or less exhaustively all work inspired by or related to the algorithm up to the time of writing (summer 1972). Our belief is that the algorithm will find application in an increasing diversity of areas. Our hope is that we can accelerate this process for the readers of this paper.

This invited paper is one of a series planned on topics of general interest—The Editor.

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II. Statement of the Problem

In its most general form, the VA may be viewed as a solution to the problem of maximum *a posteriori* probability (MAP) estimation of the state sequence of a finite-state discrete-time Markov process observed in memoryless noise. In this section we set up the problem in this generality, and then illustrate by example the different sorts of problems that can be made to fit such a model. The general approach also has the virtue of tutorial simplicity.

The underlying Markov process is characterized as follows. Time is discrete. The state x_k at time k is one of a finite number M of states m, $1 \leq m \leq M$; i.e., the state space X is simply $\{1, 2, \cdots, M\}$. Initially we shall assume that the process runs only from time 0 to time K and that the initial and final states x_0 and x_K are known; the state sequence is then represented by a finite vector $\boldsymbol{x} = (x_0, \cdots, x_K)$. We see later that extension to infinite sequences is trivial.

The process is Markov, in the sense that the probability $P(x_{k+1} | x_0, x_1, \cdots, x_k)$ of being in state x_{k+1} at time $k+1$, given all states up to time k, depends only on the state x_k at time k:

$$P(x_{k+1} | x_0, x_1, \cdots, x_k) = P(x_{k+1} | x_k).$$

The transition probabilities $P(x_{k+1} | x_k)$ may be time varying, but we do not explicitly indicate this in the notation.

It is convenient to define the *transition* ξ_k at time k as the pair of states (x_{k+1}, x_k):

$$\xi_k \triangleq (x_{k+1}, x_k).$$

We let Ξ be the (possibly time-varying) set of transitions